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The stiffness of an elastic solid with an embedded, nominally spherical inclusion subjected to a small arbitrary motion

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Abstract

The article examines the problem of translation and rotation of a nominally (slightly deformed) spherical rigid inclusion embedded into an unbounded elastic medium. To the first order in the small parameter characterizing the boundary perturbation, explicit expressions are deduced for the induced displacement field as well as for the net force and net torque required to produce the applied translation and rotation.

Keywords: Elastic solid; Nominally spherical inclusion; Translation; Rotation; Stiffness of the solid

1. Introduction

Elasticity literature abounds in solutions of problems for solids containing inclusions or cavities possessing a high degree of symmetry, e.g. those of spherical and ellipsoidal configurations (for references, see for example, Lure, 1964, 1970; Podilchuk, 1979; Teodosiu, 1982; Mura, 1987; Nemat-Nasser and Hori, 1998). For instance, theoretical calculations of the elastic fields induced in an elastic medium by static translations and rotations of spherical and ellipsoidal inclusions are given by Lure (1964, 1967, 1970), Kanwal and Sharma (1976), Walpole (1991a,b), Phan-Thien and Kim (1994), Kachanov et al. (2000) and those corresponding to time-harmonic translations and rotations of spherical inclusions by Rahman (2000). Although solutions of this type are important for synthesizing those for more complicated cases, their applicability

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is however somewhat restricted, because inclusions and cavities of such canonical shapes are atypical of the irregular shapes usually encountered in practical applications. It is Guz and Nemish (1984) who first launched a systematic investigation into a wide range of three-dimensional elasticity problems for bodies with non-canonical boundaries as well as for canonical bodies containing inclusions and flaws of non-canonical shapes. In particular, these authors solve the problems of stress concentrations near cylindrical, conical, bi-conical inclusions and cavities as well as a nearly spherical cavity in elastic solids under torsion and bi-axial remote loading. Using a regular perturbation scheme in conjunction with Papkovich–Neuber representation for the displacement vector, these authors are able to evaluate the stress concentrations near the inclusions and the flaws of the above types in closed form up to third order in the small parameter characterizing the shape perturbations. Their analyses are extremely rich in mathematical contents. The only limitation of these studies is that they are valid for only *axisymmetric* shape perturbations, i.e. perturbations that do not depend on the polar angle.[1]

In the present article, we consider the problem of a rigid inclusion in an elastic solid, whose shape deviates slightly from that of a perfect sphere (reference sphere) when the inclusion is given a small arbitrary motion with respect to the center of the reference sphere. Since any general motion of a rigid body is compounded of a motion of pure translation and one of pure rotation, the problem is therefore equivalent to two separate problems in which the first problem corresponds to the case where the inclusion is subjected to a pure translation with respect to the center of the reference sphere while the second to the case of pure rotation around the same center. In terms of the small parameter characterizing the boundary perturbation, we reduce these problems to an infinite set of problems, each satisfying the equilibrium equations and some appropriate boundary conditions on the surface of the reference sphere. We then employ Lure's general solution of elastic equilibrium equations for a spherical geometry to deduce elegant solutions for the perturbation fields. To the first order in the small parameter, we deduce explicit expressions for the induced displacement fields as well as the stiffness relations that relate the net force and net torque required to produce the applied translation and rotation of the inclusion to its geometry. In the special case where the elastic medium is an incompressible one, these results agree with those derived by Brenner (1964), for the low Reynolds number resistance of a slightly perturbed sphere to translational and rotational motions in an unbounded fluid.

The organization of the article is as follows. In Section 2, we dispose of some basic definitions and properties of spherical harmonics, which are of frequent occurrence, directly or indirectly, in the subsequent sections. In Section 3, we give precise statement of the problem and the associated boundary conditions. We then use a regular perturbation scheme to reduce the problems to an infinite set of problems in terms of the small parameter characterizing the boundary perturbation. Section 4 is devoted to the general analysis of elastic problems when displacement type boundary values are prescribed on spherical surfaces. This solution is due to Lure and is given here for the sake of completeness as well as for better understanding of the reader of the analysis to follow. Sections 5 and 6 are essentially devoted to the solutions of the perturbation problems. We have restricted the analysis up to the first order perturbation field only. It can, of course, be carried through for higher order perturbation fields using the same technique. However, the calculations beyond the first order perturbation field become very unwieldy. We deduce explicit expressions for the displacement fields for the zeroth and first order perturbation fields. In Section 7, using Betti's reciprocal theorem, we derive explicit expressions for the net forces and net torques required to produce the applied translation and rotation of the inclusion. These expressions are valid up to the first order in the small parameter characterizing the boundary perturbation. No attempt is made to establish the convergence of the perturbation solutions, which is far too complex to be investigated in a problem of this kind. However,

[1] One of the reviewers brought my attention to two more books by Guz and Nemish (1982, 1990) where solutions of some additional non-canonical three-dimensional elasticity problems are presented. I have been however not able to see these books.

as we show in Section 7, in a particular case involving the translation of a prolate spheroidal inclusion, the first order perturbation solution is within an absolute error of 6% when compared with the exact solution of the problem. The same order of accuracy can be expected from other shape perturbations.

We begin by introducing the notation that will be used in the subsequent analysis. Bold face characters are used to mean vectors. The symbol ∇ stands for the three-dimensional del operator. Second order tensors are designated by capital letters with hats. The symbol $\widehat{\mathbf{I}}$ stands for dyadic idemfactor while the symbols $\otimes$, $\bullet$ and $\times$ are meant to designate the operations of dyadic multiplication, scalar product and cross product, respectively.

2. Spherical harmonics: some basic definitions and properties

In this section, we recall some basic properties of harmonic functions and spherical harmonics that will be frequently used in the subsequent analysis. Let (R, θ, ϕ) be the corresponding spherical coordinates related to the Cartesian coordinates through the mappings

$$x_1 = R\cos\theta\sin\phi, \quad x_2 = R\sin\theta\sin\phi, \quad x_3 = R\cos\phi, \qquad 0 \leqslant R < \infty, \; 0 \leqslant \theta \leqslant 2\pi, \; 0 \leqslant \phi \leqslant \pi. \quad (1)$$

The spherical base vectors $\mathbf{e}_R$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are related to the Cartesian ones $\mathbf{e}_i$ $(i = 1,2,3)$ by the following equations:

$$\begin{Bmatrix} \mathbf{e}_R \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{Bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix}.$$

A function $F(\mathbf{x})$ is said to be harmonic in a closed region, i.e. the set of points consisting of a domain with its boundary, if it is twice differentiable and satisfies Laplace's equation, namely, $\nabla^2 F(\mathbf{x}) = 0$ at all interior points. If the region or domain is an infinite one, a supplementary condition on the behavior of the function at infinity has to be imposed (see, Kellog, 1929).

Separation of variables in Laplace's equation written in spherical coordinates shows that its solution can be represented as

$$F(\mathbf{x}) = \sum_{n=0}^{\infty} S_n$$

in the interior, and

$$F(\mathbf{x}) = \sum_{n=0}^{\infty} S_{-(n+1)}$$

in the exterior of a sphere of radius a (say) with its center coincident with the origin. Here S_n is a general solid spherical harmonic of degree n, which admits the representation

$$S_n(R,\theta,\phi) = \left(\frac{R}{a}\right)^n Y_n(\theta,\phi), \quad S_{-(n+1)}(R,\theta,\phi) = \left(\frac{a}{R}\right)^{n+1} Y_n(\theta,\phi),$$

where $n = 0, 1, 2, \ldots,$ and $Y_n(\theta, \phi)$ is the general surface spherical harmonic of degree n and is given by

$$Y_n(\theta,\phi) = \sum_{k=0}^{n} (a_k^{(n)} \cos n\theta + b_k^{(n)} \sin n\theta) P_n^{(k)}(\cos\phi), \quad b_n^{(0)} = 0.$$

Here $P_n^{(k)}(\cos\phi)$ is the associated Legendre function of the first kind, of degree n and order k, and $a_k^{(n)}, b_k^{(n)}$ $(k = 1, 2, \ldots, k)$ are $2n + 1$ arbitrary constants. For any given non-negative integral values of n, k,

$$P_n^{(k)}(\vartheta) = (1-\vartheta^2)^{k/2}\frac{\mathrm{d}^k P_n(\vartheta)}{\mathrm{d}\vartheta^k}, \quad P_n(\vartheta) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(\vartheta^2-1)^n}{\mathrm{d}\vartheta^n}, \quad P_n^{(0)}(\vartheta) = P_n(\vartheta),$$

where $P_n(\vartheta)$ is the Legendre polynomial of degree n.

Next, a vector solid spherical harmonic of degree n, $\mathbf{S}_n$, is a vector-valued function of R, θ, ϕ which allows a representation of the form

$$\mathbf{S}_n = R^n \mathbf{Y}_n(\theta, \phi),$$

in which $\mathbf{Y}_n(\theta, \phi)$ is the corresponding vector surface spherical harmonic of the same degree with the property that given any constant vector $\mathbf{a}$, $\mathbf{a} \bullet \mathbf{Y}_n(\theta, \phi)$ is a scalar surface spherical harmonic of order n. Taking, for example, $\mathbf{a} = \mathbf{e}_i$, we observe that each Cartesian component of $\mathbf{Y}_n(\theta, \phi)$ is a scalar surface spherical harmonic of degree n.

Similarly, a dyadic solid spherical harmonic of degree n, $\widehat{\mathbf{S}}_n$, is a tensor-valued function of R, θ, ϕ which allows a representation of the form

$$\widehat{\mathbf{S}}_n = R^n \widehat{\mathbf{Y}}_n(\theta, \phi),$$

in which $\widehat{\mathbf{Y}}_n(\theta, \phi)$ is the corresponding vector surface spherical harmonic of the same degree with the property that given any constant vectors $\mathbf{a}, \mathbf{b}$, $\mathbf{a} \bullet \widehat{\mathbf{Y}}_n(\theta, \phi) \bullet \mathbf{b}$ is a scalar surface spherical harmonic of degree n. Taking, for instance, $\mathbf{a} = \mathbf{e}_i$, $\mathbf{b} = \mathbf{e}_j$, we observe that each Cartesian dyadic component is a scalar surface spherical harmonic of degree n. Any polyadic solid spherical harmonic can be defined in a similar fashion.

We also recall that if Y_m, Y_n are two scalar surface spherical harmonics of different degrees m, n, then

$$\int Y_m Y_n \,\mathrm{d}\omega = 0, \tag{2}$$

where $\mathrm{d}\omega = \sin\phi \,\mathrm{d}\phi \,\mathrm{d}\theta$ and the integration is over the surface of a sphere of unit radius.

Next, recall the theorem that the value of any finite single-valued function $v(\theta, \phi)$ given over the surface of a unit sphere can be expressed, at every point of the surface at which the function is continuous, as a series of rational integral surface spherical harmonics, provided the function has only a finite number of lines and points of discontinuity and of maxima and minima and this expansion is unique (Hobson, 1955; Jeans, 1966). The expansion formula is as follows:

$$v(\theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n^{(m)}\cos m\theta + b_n^{(m)}\sin m\theta)P_n^{(m)}(\cos\phi), \quad b_n^{(0)} = 0, \tag{3}$$

where

$$\begin{aligned} a_n^{(m)} &= \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\gamma_m \int P_n^m(\cos\phi')\cos m\theta' v(\theta', \phi')\,\mathrm{d}\omega', \\ b_n^{(m)} &= \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int P_n^m(\cos\phi')\sin m\theta' v(\theta', \phi')\,\mathrm{d}\omega'. \end{aligned} \tag{4}$$

Here $\gamma_0 = 1/2$ and $\gamma_m = 1\ \forall m \geqslant 1$ and the integration is over the entire surface of a unit sphere.

For the special case where the function $v(\theta, \phi)$ is expressible in the form $v(\theta, \phi) = g_n(\mathbf{x})/R^n$ in which $g_n(\mathbf{x})$ is a finite polynomial in $\mathbf{x}$ ($\mathbf{x} \equiv (x_1, x_2, x_3)$) of degree n, it can be expanded into a series of surface spherical harmonics using a more straightforward method belonging to Gauss (see Hobson, 1955, p. 147). First, the function $g_n(\mathbf{x})$ is represented as

$$g_n(\mathbf{x}) = \sum_{i=0}^{[n/2]} R^{2i} S_{n-2i}, \tag{5}$$

where $[n/2]$ is equal to the integer part of $n/2$. The solid spherical harmonics S_{n-2i} are determined by repeated application of the Laplace operator to Eq. (5) and using the following identity:

$$\nabla^2(R^{2i}S_{n-2i}) = 2i(2n-2i+1)R^{2i-2}S_{n-2i},$$

resulting in the following simultaneous equations:

$$\begin{aligned}
\nabla^2 g_n(\mathbf{x}) &= \sum_{i=1}^{[n/2]} 2i(2n-2i+1)R^{2i-2}S_{n-2i},\\
(\nabla^2)^2 g_n(\mathbf{x}) &= \sum_{i=2}^{[n/2]} 2i(2i-2)(2n-2i+1)(2n-2i+3)R^{2i-4}S_{n-2i},\\
&\vdots\\
(\nabla^2)^k g_n(\mathbf{x}) &= 2\sum_{i=k}^{[n/2]} \frac{i!(2n-2i+1)!(n-i-k+1)!}{(i-k)!(n-i)!(2n-2i-2k+2)!}R^{2i-2k}S_{n-2i},\\
&\vdots\\
(\nabla^2)^{[n/2]} g_n(\mathbf{x}) &= \frac{2[n/2]!(n+1+\delta(n))!(1+\delta(n))!}{\{(n+\delta(n))/2\}!(2+2\delta(n))!}S_{\delta(n)},
\end{aligned} \tag{6}$$

where we have introduced the notation $\delta(2n) = 0\ \forall n = 0,1,2,\ldots$, and where $\delta(2n+1) = 1\ \forall n = 0,1,2,\ldots$ From the last equation in (6), the value of $S_{\delta(n)}$ is determined; from the preceding one the value of $S_{2+\delta(n)}$ and so on, until the value of S_n is determined. If we divide (5) by R^n, we have an expression for integral algebraic function of $\cos\theta\sin\phi$, $\sin\theta\sin\phi$, $\cos\phi$ as the sum of surface spherical harmonics, i.e.

$$v(\theta,\phi) = \frac{g_n(\mathbf{x})}{R^n} = \sum_{i=0}^{[n/2]} Y_{n-2i}, \quad S_n = R^n Y_n.$$

Similar results also hold for any polyadic integral algebraic function of $\cos\theta\sin\phi$, $\sin\theta\sin\phi$, $\cos\phi$. As a simple illustration of the above results, let us express the dyad $\mathbf{e}_R \otimes \mathbf{e}_R$ in dyadic surface spherical harmonics. Obviously, in this case, $\mathbf{v}_n(\mathbf{x}) = \hat{\mathbf{v}}_2(\mathbf{x}) = \mathbf{R}\otimes\mathbf{R}$. Then, it follows from the last equation in (6) that

$$\widehat{\mathbf{S}}_0 = \frac{1}{6}\nabla^2(\mathbf{R}\otimes\mathbf{R}) = \frac{\widehat{\mathbf{I}}}{3},$$

while from Eq. (5), we derive

$$\widehat{\mathbf{S}}_2 = \mathbf{R}\otimes\mathbf{R} - R^2\frac{\widehat{\mathbf{I}}}{3}.$$

Thus, in this case,

$$\begin{aligned}
&\mathbf{e}_R\otimes\mathbf{e}_R = \widehat{\mathbf{Y}}_2 + \widehat{\mathbf{Y}}_0,\\
&\widehat{\mathbf{Y}}_2 = \mathbf{e}_R\otimes\mathbf{e}_R - \frac{\widehat{\mathbf{I}}}{3}, \quad \widehat{\mathbf{Y}}_0 = \frac{\widehat{\mathbf{I}}}{3}.
\end{aligned} \tag{7}$$

Finally, a solid spherical harmonic of degree n is a homogeneous function of $\mathbf{x}$ of order n and as such satisfies Euler's theorem on homogeneous functions, namely,

$$\mathbf{R}\bullet\nabla S_n = nS_n.$$

Similar equations can be written for any vector, dyadic and in general for any polyadic solid spherical harmonic of degree n.

3. Statement of the problem and boundary conditions

Consider an unbounded three-dimensional space filled with a homogeneous isotropic elastic medium with Young's modulus $E > 0$ and Poisson's ratio ν $(-1 < \nu \leqslant 1/2)$, containing a rigid, nominally spherical inclusion, i.e. an inclusion whose shape deviates slightly from that of a perfectly spherical inclusion of radius a, hereafter called the reference sphere. The inclusion is given a small general motion in any way with respect to the center of the reference sphere. It is well known that any general motion of a rigid body is compounded of a motion of translation and one of rotation. In view of this observation as well as the linear nature of the response of the medium, it follows that there will be no loss in generality if we treat these cases separately. Thus, let us assume that the inclusion is given a small translation $\mathbf{u}_0$ and a small rotation $\boldsymbol{\omega}$ with regard to the center of the reference sphere along some arbitrary directions. We introduce a Cartesian coordinate system with its origin at the center of the reference sphere. Then, the surface of the nominally spherical inclusion may be described by an equation of the following form

$$R = a[1 + \varepsilon f(\theta, \phi)], \tag{8}$$

in which $\varepsilon f(\theta, \phi)$ is an arbitrary function such that $\max|\varepsilon f(\theta, \phi)| \ll 1$. In what follows the three-dimensional space with the nominally spherical inclusion deleted will be denoted by Ω, while that with the reference sphere deleted by Ω_0. The symbols $\partial\Omega$, $\partial\Omega_0$ are used to designate the surface of the nominally spherical inclusion and that of the reference sphere, respectively.

In the absence of body forces, the field equations characterizing the equilibrium of a linearly elastic, homogeneous isotropic solid are given by

$$\begin{aligned} &2\widehat{\boldsymbol{\chi}} = \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}, \\ &\hat{\boldsymbol{\sigma}} = 2\mu\left(\frac{\nu}{1-2\nu}\widehat{\mathbf{I}}\nabla \bullet \mathbf{u} + \widehat{\boldsymbol{\chi}}\right), \\ &\nabla \bullet \hat{\boldsymbol{\sigma}} = 0, \end{aligned} \tag{9}$$

in which $\mathbf{u}$ is the displacement vector, $(\nabla\mathbf{u})^{\mathrm{T}}$ is the transpose of $\nabla\mathbf{u}$, $\widehat{\boldsymbol{\chi}}$ is the strain tensor and $\hat{\boldsymbol{\sigma}}$ is the corresponding stress tensor. Provided the displacement vector is at least twice differentiable, elimination of the stresses among equations in (9) leads to the displacement equations of equilibrium

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla\nabla \bullet \mathbf{u} = 0 \quad \text{in } \Omega. \tag{10}$$

The solution of Eq. (10) is subject to the following boundary condition

$$\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{R} \quad \text{on } \partial\Omega. \tag{11}$$

Furthermore, the solution must satisfy the condition at infinity requiring that $\mathbf{u}$ be at least of $\mathrm{O}(R^{-1})$ as $R \to \infty$ (uniformly in θ and ϕ).

The net force and the net torque exerted by the inclusion on the medium are given by the formulae

$$\mathbf{P} = \int_{\partial\Omega} \mathbf{n} \bullet \hat{\boldsymbol{\sigma}}\,\mathrm{d}\Omega, \quad \mathbf{M} = \int_{\partial\Omega} \mathbf{R} \times (\mathbf{n} \bullet \hat{\boldsymbol{\sigma}})\,\mathrm{d}\Omega, \tag{12}$$

where $\mathbf{n}$ is the surface normal directed into the inclusion.

The presence of the small quantity ε induces the idea to represent the displacement vector as

$$\mathbf{u} = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}^{(i)} \quad \text{in } \Omega. \tag{13}$$

Then putting (13) into (10) and equating the coefficients in the like powers of ε in the resulting equations, we see that each individual perturbation $\mathbf{u}^{(i)}$ satisfies an equation of the form (10), namely,

$$\nabla^2\mathbf{u}^{(i)} + \frac{1}{1-2v}\nabla\nabla \bullet \mathbf{u}^{(i)} = 0 \quad \text{in } \Omega. \tag{14}$$

Considering the case of translation first, the solution of Eqs. (14) must satisfy the following boundary condition on the surface of the inclusion

$$\sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}^{(i)} = \mathbf{u}_0 \quad \text{on } \partial\Omega, \tag{15}$$

and the condition at infinity

$$\mathbf{u}^{(i)} = \mathrm{O}(R^{-1}) \quad \text{as } R \to \infty \text{ (uniformly in } \theta \text{ and } \phi). \tag{16}$$

Expanding the function $\mathbf{u}^{(i)}$ in Taylor's series around $R = a$, we have

$$\mathbf{u}^{(i)} = \sum_{j=0}^{\infty} \frac{1}{j!}(R-a)^j \frac{\partial^i \mathbf{u}^{(i)}}{\partial R}\bigg|_{R=a} = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} a^j f^j(\theta,\phi) \frac{\partial^i \mathbf{u}^{(i)}}{\partial R^j}\bigg|_{R=a}. \tag{17}$$

Putting (17) into (15), we obtain

$$\sum_{i=0}^{\infty} \varepsilon^i \sum_{j=0}^{\infty} \varepsilon^j (af(\theta,\phi))^j \frac{\partial^j \mathbf{u}^{(i)}}{\partial R^j}\bigg|_{R=a} = \mathbf{u}_0 \quad \text{on } \partial\Omega,$$

which can be rewritten as

$$\sum_{i=0}^{\infty} \varepsilon^i \left[\mathbf{u}^{(i)} + \sum_{j=1}^{i} \frac{1}{j!}(af(\theta,\phi))^j \frac{\partial^j \mathbf{u}^{(i-j)}}{\partial R^j}\bigg|_{R=a}\right] = \mathbf{u}_0 \quad \text{on } \partial\Omega. \tag{18}$$

Now equating the like powers of ε on both sides of Eq. (18), we have

$$\begin{aligned} &\mathbf{u}^{(0)} = \mathbf{u}_0 \quad \text{on } \partial\Omega_0, \\ &\mathbf{u}^{(i)} = -\sum_{j=1}^{i} \frac{1}{j!} a^j f^j(\theta,\phi) \frac{\partial^j \mathbf{u}^{(i-j)}}{\partial R^j} \quad \text{on } \partial\Omega_0 \quad \forall i = 1,2,3,\ldots \end{aligned} \tag{19}$$

Similarly, for the case of pure rotation of the nominally spherical inclusion, the boundary conditions to be satisfied on the surface of the reference sphere are

$$\begin{aligned} &\mathbf{u}^{(0)} = \boldsymbol{\omega} \times \mathbf{R} \quad \text{on } \partial\Omega_0, \\ &\mathbf{u}^{(1)} = af(\theta,\phi)\left(\boldsymbol{\omega} \times \mathbf{e}_R - \frac{\partial \mathbf{u}^{(0)}}{\partial R}\right) \quad \text{on } \partial\Omega_0, \\ &\mathbf{u}^{(i)} = -\sum_{j=1}^{i} \frac{1}{j!} a^j f^j(\theta,\phi) \frac{\partial^j \mathbf{u}^{(i-j)}}{\partial R^j} \quad \text{on } \partial\Omega_0 \quad \forall i = 2,3,\ldots \end{aligned} \tag{20}$$

Thus, the problems are reduced to those of finding the solutions of the system of Eq. (14) satisfying the boundary conditions (19) and (20) prescribed on the surface of the reference sphere and the conditions at infinity.

4. General analysis

It is well known (see, for instance, Lure, 1964) that in the absence of body forces, the displacement vector becomes a biharmonic vector. In this case, as shown by Trefftz, the displacement vector may be represented in the form

$$\mathbf{u} = \mathbf{U} + (R^2 - a^2)\nabla\Psi, \tag{21}$$

where Ψ, $\mathbf{U}$ are respectively a harmonic scalar and a harmonic vector. This representation is especially suitable for cases where boundary conditions are prescribed on spherical surfaces as is the present case. It can be shown by putting (21) into the equations of equilibrium (10) that

$$(1 - 2v)\Psi + (3 - 4v)\mathbf{R} \bullet \nabla\Psi + \frac{1}{2}\nabla \bullet \mathbf{U} = 0. \tag{22}$$

It is assumed that the displacement vector given on the surface of the sphere meets the requirements outlined in Section 2 and as such can be expanded into a series of surface spherical harmonics, namely,

$$\mathbf{u}|_{R=a} = \mathbf{U}|_{R=a} = \sum_{k=0}^{\infty} \mathbf{Y}_k(\theta, \phi), \tag{23}$$

where $\mathbf{Y}_k(\theta, \phi)$ is a vector surface spherical harmonic of degree k.

Thus for the external problem ($R > a$) we have

$$\mathbf{U} = \sum_{k=0}^{\infty} \mathbf{U}_{-(k+1)} = \sum_{k=0}^{\infty} \left(\frac{a}{R}\right)^{k+1} \mathbf{Y}_k(\theta, \phi), \tag{24}$$

where $\mathbf{U}_{-(k+1)}$ are the homogeneous harmonic vectors of degree $-k - 1$. Similarly, the harmonic scalar Ψ is also expanded into surface spherical harmonics, namely,

$$\Psi = \sum_{k=0}^{\infty} \Psi_{-(k+1)} = \sum_{k=0}^{\infty} \left(\frac{a}{R}\right)^{k+1} Z_{-(k+1)}(\theta, \phi). \tag{25}$$

Putting (24) and (25) into (22), the following equation is obtained:

$$(1 - 2v)\Psi_{-(k+1)} - (3 - 4v)(k + 2)\Psi_{-(k+1)} + \frac{1}{2}\nabla \bullet \mathbf{U}_{-(k+1)} = 0, \tag{26}$$

whence it follows that

$$\Psi_{-(k+1)} = \frac{\nabla \bullet \mathbf{U}_{-(k+1)}}{2[3k + 5 - 2v(2k + 3)]}. \tag{27}$$

Therefore, the solution of the external problem is given by

$$\mathbf{u} = \sum_{k=0}^{\infty} \left[\mathbf{U}_{-(k+1)} + \frac{1}{2}(R^2 - a^2)\frac{\nabla\nabla \bullet \mathbf{U}_{-(k+1)}}{3k + 5 - 2v(2k + 3)}\right] \quad \text{in } \Omega. \tag{28}$$

This is the general solution of the problem given in Lure (1970). It is important to note that since each perturbation field satisfies equations that are exactly of the same form as those for elastic equilibrium (see Eq. (14)), the representation (28) is therefore valid for all perturbation problems. We will therefore use Eq. (28) to deduce solutions for the different perturbation fields. In view of greater simplicity, we will commence with the case of pure rotation.

5. Rotation of a nominally spherical inclusion

In this case, the boundary conditions of the problem are (limiting up to only first order perturbation):

$$\mathbf{u}^{(0)} = \boldsymbol{\omega} \times \mathbf{R} \quad \text{on } \partial\Omega_0,$$
$$\mathbf{u}^{(1)} = af(\theta,\phi)\left(\boldsymbol{\omega} \times \mathbf{e}_R - \frac{\partial \mathbf{u}^{(0)}}{\partial R}\right) \quad \text{on } \partial\Omega_0. \tag{29}$$

Solution of the zeroth order perturbation is straightforward, because the right hand side of the first equation in (29) is already a surface vector spherical harmonic of degree one, namely

$$\mathbf{Y}_1 = a\boldsymbol{\omega} \times \mathbf{e}_R.$$

Hence, the only surviving member of the sequence $\{\mathbf{U}_{-(k+1)}\}$ in Eq. (28) is $\mathbf{U}_{-2}$ which is given by

$$\mathbf{U}_{-2} = \left(\frac{a}{R}\right)^2 \mathbf{Y}_1 = \frac{a^3}{R^2}\boldsymbol{\omega} \times \mathbf{e}_R. \tag{30}$$

Putting (30) into the general solution (28), we obtain

$$\mathbf{u}^{(0)} = \mathbf{U}_{-2} + \frac{1}{2}(R^2 - a^2)\frac{\nabla\nabla \bullet \mathbf{U}_{-2}}{8 - 10\nu} \quad \text{in } \Omega_0. \tag{31}$$

However, $\nabla \bullet \mathbf{U}_{-2} = 0$, and so Eq. (31) takes the following final form:

$$\mathbf{u}^{(0)} = \frac{a^3}{R^2}\boldsymbol{\omega} \times \mathbf{e}_R \quad \text{in } \Omega_0. \tag{32}$$

The corresponding stress-vector corresponding to a surface with the outward normal **n** can be shown to be

$$\mathbf{n} \bullet \boldsymbol{\sigma}^{(0)} = 3\mu\frac{a^3}{R^3}(\boldsymbol{\omega} \times \mathbf{n} - 2\boldsymbol{\omega} \times \mathbf{e}_R \otimes \mathbf{n} \bullet \mathbf{e}_R + \mathbf{n} \times \mathbf{e}_R \otimes \boldsymbol{\omega} \bullet \mathbf{e}_R). \tag{33}$$

For our problem, $\mathbf{n} = -\mathbf{e}_R$, and therefore, Eq. (33) gives

$$\mathbf{n} \bullet \hat{\boldsymbol{\sigma}}^{(0)} = -3\mu\mathbf{e}_R \times \boldsymbol{\omega} \quad \text{on } \partial\Omega_0. \tag{34}$$

Thus, the net force exerted by the inclusion on the medium is

$$\mathbf{P}^{(0)} = -\int_{\partial\Omega_0} \mathbf{e}_R \bullet \hat{\boldsymbol{\sigma}}^{(0)}\,\mathrm{d}\tau = -3\mu\int_{\partial\Omega_0} \mathbf{e}_R \times \boldsymbol{\omega}\,\mathrm{d}\tau = -3\mu\left(\int_{\partial\Omega_0} \mathbf{e}_R\,\mathrm{d}\tau\right) \times \boldsymbol{\omega} = \mathbf{0}, \tag{35}$$

where **0** is a null vector. Therefore, the inclusion does not exert any force on the medium.

On the other hand, the net torque required to produce the rotation $\boldsymbol{\omega}$ is

$$\mathbf{M}^{(0)} = \int_{\partial\Omega_0} \mathbf{R} \times (\mathbf{n} \bullet \hat{\boldsymbol{\sigma}}^{(0)})\,\mathrm{d}\tau = -3\mu\int_{\partial\Omega_0} \mathbf{R} \times (\mathbf{e}_R \times \boldsymbol{\omega})\,\mathrm{d}\tau. \tag{36}$$

However, $\mathbf{R} \times (\mathbf{e}_R \times \boldsymbol{\omega}) = \mathbf{e}_R \otimes \mathbf{R} \bullet \boldsymbol{\omega} - R\boldsymbol{\omega}$. Therefore, Eq. (36) takes the form

$$\mathbf{M}^{(0)} = -3\mu a\boldsymbol{\omega} \bullet \int_{\partial\Omega_0} \mathbf{e}_R \otimes \mathbf{e}_R\,\mathrm{d}\tau + 3\mu a\boldsymbol{\omega}\int_{\partial\Omega_0} \mathrm{d}\tau. \tag{37}$$

Using the expansion of the dyad $\mathbf{e}_R \otimes \mathbf{e}_R$ into surface spherical harmonics (see Eqs. (7)) and invoking the orthogonality property of the surface spherical harmonics of different orders over the surface of a unit sphere, we find that

$$\mathbf{M}^{(0)} = -\mu a\boldsymbol{\omega} \bullet \widehat{\mathbf{I}}\int_{\partial\Omega_0} \mathrm{d}\tau + 3\mu a\boldsymbol{\omega}\int_{\partial\Omega_0} \mathrm{d}\tau = 2\mu a\boldsymbol{\omega}\int_{\partial\Omega_0} \mathrm{d}\tau = 8\pi\mu a^3\boldsymbol{\omega}. \tag{38}$$

This is the solution of Lure, of the problem of a perfectly spherical rigid inclusion in bonded contact with an elastic solid when the former is subjected to a small constant rotation in an arbitrary direction.

We rewrite Eq. (38) in the form:

$$\mathbf{M}^{(0)} = \widehat{\mathbf{\Lambda}}^{(0)} \bullet \boldsymbol{\omega}, \tag{39}$$

where

$$\widehat{\mathbf{\Lambda}}^{(0)} = 8\pi\mu a^3 \widehat{\mathbf{I}}. \tag{40}$$

We call the symmetric tensor $\widehat{\mathbf{\Lambda}}^{(0)}$ the rotational stiffness tensor. Physically, the component $\widehat{\Lambda}_{ij}$ characterizes the ith component of the torque required to produce a unit rotation of the inclusion around the x_j-axis.

This completes the solution of the zeroth order perturbation field.

5.1. Solution of the first order perturbation field

We now proceed to the solution corresponding to the first order perturbation field. The boundary condition to be satisfied is

$$\mathbf{u}^{(1)} = af(\theta, \phi)\left(\boldsymbol{\omega} \times \mathbf{e}_R - \frac{\partial \mathbf{u}^{(0)}}{\partial R}\right) \quad \text{on } \partial\Omega_0. \tag{41}$$

Assuming that $f(\theta, \phi)$ is a sufficiently smooth function to allow for a uniformly convergent in θ and ϕ series-expansion like

$$f(\theta, \phi) = \sum_{k=0}^{\infty} f_k(\theta, \phi), \tag{42}$$

where $f_k(\theta, \phi)$ is a surface spherical harmonic of degree k, we rewrite Eq. (41) as

$$\mathbf{u}^{(1)} = 3a \sum_{k=0}^{\infty} \boldsymbol{\omega} \times \mathbf{e}_R f_k(\theta, \phi) \quad \text{on } \partial\Omega_0. \tag{43}$$

The problem now is to expand (43) into a series of surface spherical harmonics. To do this, we can use Eqs. (3) and (4) for expanding any arbitrary function of θ, ϕ into surface spherical harmonics. However, for our problem, it is more expedient to follow a different route based on a rather ad hoc method. The presence of the term $\mathbf{e}_R f_k$ in (43) naturally induces the idea to consider the gradient of a function like $R^m f_k$. For instance,

$$\nabla(R^m f_k) = mR^{m-1}\mathbf{e}_R f_k + R^m \nabla f_k,$$

so that the function

$$R^{1-m}\nabla(R^m f_k) = m\mathbf{e}_R f_k + R\nabla f_k \tag{44}$$

is a homogeneous function of order zero, but not necessarily a surface spherical harmonic. We now pose the question: for what values of m, is $R^{1-m}\nabla(R^m f_k)$ a surface spherical harmonic? Assuming that it is a surface spherical harmonic of degree p, the question is how m and p are related to k. Since, by hypothesis it is a surface spherical harmonic of degree p, the relevant solid spherical harmonic must satisfy Laplace's equation, namely,

$$\begin{aligned}\nabla^2(R^{p+1-m}\nabla(R^m f_k)) = {} & \nabla^2(R^{1-m+p})\nabla(R^m f_k) + R^{1-m+p}\nabla(\nabla^2(R^m f_k)) \\ & + 2(1-m+p)R^{p-m-1}\mathbf{R} \bullet \nabla\nabla(R^m f_k) = 0.\end{aligned}$$

The above expression, after some simple vector manipulations, can be rewritten as

$$\nabla^2(R^{p+1-m}\nabla(R^m f_k)) = [2m - k(k+1) + p(p+1)]R^{p-k-1}\nabla(R^k f_k) + (m-k)[p(p+1) - (k+1)(k+2)]R^{p-3} f_k \mathbf{R} = 0. \tag{45}$$

For Eq. (45) to be identically zero for all values of $\mathbf{R}$, R and f_k, it is necessary that

$$\begin{aligned} &[2m - k(k+1) + p(p+1)] = 0,\\ &(m-k)[p(p+1) - (k+1)(k+2)] = 0. \end{aligned} \tag{46}$$

From the second equation in (46), it follows that

$$m = k \quad \text{or} \quad m = -k-1. \tag{47}$$

On the other hand, the second equation in (46) implies that if $m = k$, then $p = -k$ or $p = k-1$ while if $m = -k-1$, then $p = k+1$ or $p = -k-2$. Thus, there could be two distinct scenarios, namely

$\mathbf{C}_{k-1} \equiv R^{1-k}\nabla(R^k f_k)$ which is a surface spherical harmonic of degree $k-1$. Corresponding to $\mathbf{C}_{k-1}$, there are two solid spherical harmonics, one of degree $-k$ and the other of degree $k-1$;[2]
$\mathbf{D}_{k+1} \equiv R^{k+2}\nabla(R^{-1-k} f_k)$ which is a surface spherical harmonic of degree $k+1$. Corresponding to $\mathbf{D}_{k+1}$, there are two solid spherical harmonics, one of degree $-k-2$ and the other of degree $k+1$.

Of course, for our exterior problem, the solid spherical harmonics corresponding to the predominantly positive degrees, namely, $k-1$ and $k+1$, must be discarded.

We thus have

$$\begin{aligned} \mathbf{C}_{k-1} &\equiv R^{1-k}\nabla(R^k f_k) = k\mathbf{e}_R f_k + R\nabla f_k,\\ \mathbf{D}_{k+1} &\equiv R^{k+2}\nabla(R^{-1-k} f_k) = -(k+1)\mathbf{e}_R f_k + R\nabla f_k. \end{aligned} \tag{48}$$

Solving equations in (48) for $\mathbf{e}_R f_k$, we obtain

$$\mathbf{e}_R f_k = \frac{1}{2k+1}(\mathbf{C}_{k-1} - \mathbf{D}_{k+1}). \tag{49}$$

Putting (49) into (43), we have

$$\mathbf{u}^{(1)} = 3a\sum_{k=0}^{\infty} f_k \boldsymbol{\omega} \times \mathbf{e}_R = 3a\sum_{k=0}^{\infty} \frac{1}{2k+1}\boldsymbol{\omega} \times (\mathbf{C}_{k-1} - \mathbf{D}_{k+1}) \quad \text{on } \partial\Omega_0. \tag{50}$$

Thus, with (50), the boundary data (41) is expanded into surface spherical harmonics and the solution of the first order perturbation field is given by

$$\mathbf{u}^{(1)} = 3a\sum_{k=0}^{\infty} \frac{1}{2k+1}\boldsymbol{\omega} \times \left[\left(\frac{a}{R}\right)^k \mathbf{C}_{k-1} - \left(\frac{a}{R}\right)^{k+2}\mathbf{D}_{k+1} + \frac{1}{2}(R^2 - a^2)\frac{1}{3k+2-2\nu(2k+1)}\nabla\nabla \bullet \left\{\left(\frac{a}{R}\right)^k \mathbf{C}_{k-1}\right\} - \frac{1}{2}(R^2 - a^2)\frac{1}{3k+8-2\nu(2k+5)}\nabla\nabla \bullet \left\{\left(\frac{a}{R}\right)^{k+2}\mathbf{D}_{k+1}\right\}\right]$$

in Ω_0.

[2] This is in accord with a well-known result in the theory of spherical harmonics, that says that corresponding to every surface spherical harmonic of degree k, there correspond two solid spherical harmonics, one of degree k and the other of degree $-k-1$.

Thus, to O(ε), the solution of the problem is

$$\mathbf{u} = \frac{a^3}{R^2}\boldsymbol{\omega}\times\mathbf{e}_R + 3\varepsilon a\sum_{k=0}^{\infty}\frac{1}{2k+1}\boldsymbol{\omega}\times\left[\left(\frac{a}{R}\right)^k\mathbf{C}_{k-1} - \left(\frac{a}{R}\right)^{k+2}\mathbf{D}_{k+1} + \frac{1}{2}(R^2-a^2)\frac{1}{3k+2-2v(2k+1)}\nabla\nabla\right.$$
$$\left.\bullet\left\{\left(\frac{a}{R}\right)^k\mathbf{C}_{k-1}\right\} - \frac{1}{2}(R^2-a^2)\frac{1}{3k+8-2v(2k+5)}\nabla\nabla\bullet\left\{\left(\frac{a}{R}\right)^{k+2}\mathbf{D}_{k+1}\right\}\right] + \mathrm{O}(\varepsilon^2) \tag{51}$$

in Ω.

The correctness of this solution can be immediately checked against the solution of a rather a trivial case. Specifically, consider the case where the spherical inclusion is given a uniform perturbation εa along the outward normal. In this case,

$$f_k = \begin{Bmatrix} 1, & k=0 \\ 0, & k \geqslant 1 \end{Bmatrix},$$

and so

$$\begin{aligned} &\mathbf{C}_{k-1} = 0 \quad \forall k = 0,1,2,\ldots \\ &\mathbf{D}_{k+1} = \begin{Bmatrix} -\mathbf{e}_R, & k=0 \\ 0, & k=1,2,3,\ldots \end{Bmatrix}. \end{aligned} \tag{52}$$

Putting (52) into (51), we obtain

$$\mathbf{u} = \frac{a^3(1+3\varepsilon)}{R^2}\boldsymbol{\omega}\times\mathbf{e}_R + \mathrm{O}(\varepsilon^2) \quad \text{in } \Omega. \tag{53}$$

Note that as a result of this perturbation, the perfectly spherical inclusion of radius a again becomes one of radius $a(1+\varepsilon)$ and hence the exact solution of the problem, as per Eq. (32), is given by

$$\mathbf{u} = \frac{a^3(1+\varepsilon)^3}{R^2}\boldsymbol{\omega}\times\mathbf{e}_R \quad \text{in } \Omega,$$

which, to first order in ε, agrees with (53). This simple test renders credence to the correctness of the solution (51).

6. The translating nominally spherical inclusion

We now proceed to the solution of the problem of translation of the nominally spherical inclusion. Limiting up to the first order perturbation field, the boundary conditions to be satisfied on the surface of the reference sphere are

$$\begin{aligned} &\mathbf{u}^{(0)} = \mathbf{u}_0 \quad \text{on } \partial\Omega_0, \\ &\mathbf{u}^{(1)} = \frac{3}{5-6v}[2(1-v)\widehat{\mathbf{I}} - \mathbf{e}_R\otimes\mathbf{e}_R]\bullet\mathbf{u}_0 f(\theta,\phi) \quad \text{on } \partial\Omega_0. \end{aligned} \tag{54}$$

Focusing first on the zeroth order problem, we note that $\mathbf{Y}_0 = \mathbf{u}_0$, and hence the only surviving member of the sequence $\{\mathbf{U}_{-(k+1)}\}$ in (28) is $\mathbf{U}_{-1}$ which is given by

$$\mathbf{U}_{-1} = \frac{a}{R}\mathbf{u}_0.$$

Therefore, the solution of the zeroth order perturbation field is

$$\mathbf{u}^{(0)} = \mathbf{U}_{-1} + \frac{1}{2}(R^2-a^2)\frac{\nabla\nabla\bullet\mathbf{U}_{-1}}{5-6v} = \frac{a}{R}\mathbf{u}_0 + \frac{R^2-a^2}{2(5-6v)}\frac{a}{R^3}(3\mathbf{e}_R\otimes\mathbf{e}_R - \widehat{\mathbf{I}})\bullet\mathbf{u}_0 \tag{55}$$

in Ω_0.

It can be easily shown that the stress-vector corresponding to the surface $R = a$ is

$$\mathbf{n} \bullet \hat{\boldsymbol{\sigma}}^{(0)} = -\mathbf{e}_R \bullet \hat{\boldsymbol{\sigma}}^{(0)} = \frac{6\mu(1-v)}{(5-6v)a}\mathbf{u}_0 \quad \text{on } \partial\Omega_0. \tag{56}$$

Thus, the total force exerted by the reference spherical inclusion on the medium due to a constant translation $\mathbf{u}_0$ is given by

$$\mathbf{P}^{(0)} = \int_{\partial\Omega_0} \mathbf{n} \bullet \hat{\boldsymbol{\sigma}}^{(0)}\,\mathrm{d}\tau = \frac{24\pi\mu(1-v)}{5-6v}a\mathbf{u}_0. \tag{57}$$

This is the solution of Lure, of the problem of a perfectly spherical inclusion in bonded contact with an elastic solid when the former is subjected to a small constant translation in an arbitrary direction.

The inclusion does not exert any torque with respect to the origin since

$$\mathbf{M}^{(0)} = \int_{\partial\Omega_0} \mathbf{R} \times \mathbf{P}^{(0)}\,\mathrm{d}\tau = \frac{24\pi\mu(1-v)a^2}{5-6v}\left(\int_{\partial\Omega_0} \mathbf{e}_R\,\mathrm{d}s\right) \times \mathbf{u}_0 = \mathbf{0}. \tag{58}$$

We write the expression (57) as

$$\mathbf{P}^{(0)} = \widehat{\boldsymbol{\Gamma}}^{(0)} \bullet \mathbf{u}_0, \tag{59}$$

where

$$\widehat{\boldsymbol{\Gamma}}^{(0)} = \frac{24\pi\mu(1-v)a}{5-6v}\widehat{\mathbf{I}}. \tag{60}$$

We call $\widehat{\boldsymbol{\Gamma}}^{(0)}$ the translational stiffness tensor. Essentially, $\Gamma_{ij}^{(0)}$ is the ith component of the force required to produce a unit translation of the reference spherical inclusion along x_j-axis.

6.1. Solution of the first order perturbation field

We now proceed to the solution for the first order perturbation field. The boundary condition to be satisfied on the spherical boundary $R = a$ is

$$\begin{aligned}\mathbf{u}^{(1)} &= \frac{3}{5-6v}[2(1-v)\widehat{\mathbf{I}} - \mathbf{e}_R \otimes \mathbf{e}_R] \bullet \mathbf{u}_0 f(\theta,\phi) \\ &= \frac{3}{5-6v}\sum_{k=0}^{\infty}[2(1-v)\widehat{\mathbf{I}}f_k(\theta,\phi) - \mathbf{e}_R \otimes \mathbf{e}_R f_k(\theta,\phi)] \bullet \mathbf{u}_0 \quad \text{on } \partial\Omega_0.\end{aligned} \tag{61}$$

The problem now consists in expanding the right hand side of Eq. (61) into a series of surface spherical harmonics. Notice that the first term in Eq. (61), namely $\widehat{\mathbf{I}} \bullet \mathbf{u}_0 f_k(\theta,\phi)$ is already a surface spherical harmonic of degree k. The problem then reduces to that of expanding the term $\mathbf{e}_R \otimes \mathbf{e}_R f_k(\theta,\phi)$ into a series of dyadic surface spherical harmonics. To this end, we again adopt a technique similar to that developed in the previous section. In particular, we note that the presence of the dyad $\mathbf{e}_R \otimes \mathbf{e}_R$ is suggestive that we look closely into quantities of the form $\nabla(R^{m_1}\mathbf{C}_{k-1}), \nabla(R^{m_2}\mathbf{D}_{k+1})$; we have

$$\begin{aligned}\nabla(R^{m_1}\mathbf{C}_{k-1}) &= k(m_1-1)R^{m_1-1}\mathbf{e}_R \otimes \mathbf{e}_R f_k + kR^{m_1-1}\widehat{\mathbf{I}}f_k + (m_1+1)R^{m_1}\mathbf{e}_R \otimes \nabla f_k \\ &\quad + kR^{m_1}\nabla f_k \otimes \mathbf{e}_R + R^{m_1+1}\nabla \otimes \nabla f_k, \\ \nabla(R^{m_2}\mathbf{D}_{k+1}) &= (k+1)(1-m_2)R^{m_2-1}\mathbf{e}_R \otimes \mathbf{e}_R f_k - (k+1)R^{m_2-1}\widehat{\mathbf{I}}f_k + (m_2+1)R^{m_2}\mathbf{e}_R \otimes \nabla f_k \\ &\quad - (k+1)R^{m_2}\nabla f_k \otimes \mathbf{e}_R + R^{m_2+1}\nabla \otimes \nabla f_k,\end{aligned}$$

so that the functions

$$\begin{aligned} R^{1-m_1}\nabla(R^{m_1}\mathbf{C}_{k-1}) &= k(m_1-1)\mathbf{e}_R\otimes\mathbf{e}_R f_k + k\widehat{\mathbf{I}}f_k + (m_1+1)R\mathbf{e}_R\otimes\nabla f_k \\ &\quad + kR\nabla f_k\otimes\mathbf{e}_R + R^2\nabla\otimes\nabla f_k, \\ R^{1-m_2}\nabla(R^{m_2}\mathbf{D}_{k+1}) &= (k+1)(1-m_2)\mathbf{e}_R\otimes\mathbf{e}_R f_k - (k+1)\widehat{\mathbf{I}}f_k + (m_2+1)R\mathbf{e}_R\otimes\nabla f_k \\ &\quad - (k+1)R\nabla f_k\otimes\mathbf{e}_R + R^2\nabla\otimes\nabla f_k, \end{aligned} \tag{62}$$

are homogeneous of degree zero. Requiring that they are surface spherical harmonics of degrees n_1 and n_2, respectively, the question is how m_i, n_i $(i=1,2)$ are related to k. Since by hypothesis they are surface spherical harmonics of degrees n_1 and n_2, the corresponding solid spherical harmonics must satisfy Laplace's equation, namely,

$$\nabla^2\{R^{1-m_1+n_1}\nabla(R^{m_1}\mathbf{C}_{k-1})\} = 0, \quad \nabla^2\{R^{1-m_2+n_2}\nabla(R^{m_2}\mathbf{D}_{k+1})\} = 0.$$

These equations yield

$$(m_1+n_1)(n_1-m_1+1) = 0, \quad (m_2+n_2)(n_2-m_2+1) = 0. \tag{63}$$

Take, for instance, the first equation in (63). To simplify the analysis, let us put

$$m_1 = k-1 \quad \text{or} \quad m_1 = -k. \tag{64}$$

If $m_1 = k-1$, n_1 is equal to either $-(k-1)$ or $k-2$. On the other hand, when $m_1 = -k$, n_1 is equal to either k or $-k-1$. Accordingly, two different dyadic surface spherical harmonics emerge, namely,

$R^{2-k}\nabla(R^{k-1}\mathbf{C}_{k-1})$ which is a surface spherical harmonic of degree $k-2$; corresponding to this surface spherical harmonics are two solid spherical harmonics, one of degree $-(k-1)$ and the other of degree $k-2$;
$R^{k+1}\nabla(R^{-k}\mathbf{C}_{k-1})$ which is a surface spherical harmonic of degree k; corresponding to this surface spherical harmonics are two solid spherical harmonics, one of degree k and the other of degree $-k-1$.

Similarly, putting $m_2 = k+1$, $-k-2$, we obtain two more surface spherical harmonics, namely,

$R^{-k}\nabla(R^{k+1}\mathbf{D}_{k+1})$ which is a surface spherical harmonic of degree k; corresponding to this surface spherical harmonics are two solid spherical harmonics, one of degree $-(k+1)$ and the other of degree k;
$R^{k+3}\nabla(R^{-k-2}\mathbf{D}_{k+1})$ which is a surface spherical harmonic of degree $k+2$ and corresponding to this surface spherical harmonics are two solid spherical harmonics, one of degree $k+2$ and the other of degree $-k-3$.

Evidently, for our exterior problem, the solid spherical harmonics corresponding to the predominantly positive degrees, namely, $k-2$, k, $k+2$, should be abandoned.

We thus have

$$\begin{aligned} \widehat{\mathbf{E}}_{k-2} &\equiv R^{2-k}\nabla(R^{k-1}\mathbf{C}_{k-1}) = k(k-2)\mathbf{e}_R\otimes\mathbf{e}_R f_k + k\widehat{\mathbf{I}}f_k + kR\mathbf{e}_R\otimes\nabla f_k + kR\nabla f_k\otimes\mathbf{e}_R + R^2\nabla\otimes\nabla f_k, \\ \widehat{\mathbf{F}}_k &\equiv R^{k+1}\nabla(R^{-k}\mathbf{C}_{k-1}) = -k(k+1)\mathbf{e}_R\otimes\mathbf{e}_R f_k + k\widehat{\mathbf{I}}f_k - (k-1)R\mathbf{e}_R\otimes\nabla f_k \\ &\quad + kR\nabla f_k\otimes\mathbf{e}_R + R^2\nabla\otimes\nabla f_k, \\ \widehat{\mathbf{G}}_k &\equiv R^{-k}\nabla(R^{k+1}\mathbf{D}_{k+1}) = -k(k+1)\mathbf{e}_R\otimes\mathbf{e}_R f_k - (k+1)\widehat{\mathbf{I}}f_k + (k+2)R\mathbf{e}_R\otimes\nabla f_k \\ &\quad - (k+1)R\nabla f_k\otimes\mathbf{e}_R + R^2\nabla\otimes\nabla f_k, \\ \widehat{\mathbf{H}}_{k+2} &\equiv R^{k+3}\nabla(R^{-k-2}\mathbf{D}_{k+1}) = (k+1)(k+3)\mathbf{e}_R\otimes\mathbf{e}_R f_k - (k+1)\widehat{\mathbf{I}}f_k - (k+1)R\mathbf{e}_R\otimes\nabla f_k \\ &\quad - (k+1)R\nabla f_k\otimes\mathbf{e}_R + R^2\nabla\otimes\nabla f_k. \end{aligned} \tag{65}$$

Denoting $\widehat{\mathbf{T}}_k = \widehat{\mathbf{I}} f_k$ which is, of course, a dyadic surface spherical harmonic of degree k, we rewrite Eqs. (65) as

$$\begin{aligned}
&k(k-2)\mathbf{e}_R \otimes \mathbf{e}_R f_k + kR\mathbf{e}_R \otimes \nabla f_k + kR\nabla f_k \otimes \mathbf{e}_R + R^2\nabla \otimes \nabla f_k = \widehat{\mathbf{E}}_{k-2} - k\widehat{\mathbf{T}}_k,\\
&-k(k+1)\mathbf{e}_R \otimes \mathbf{e}_R f_k - (k-1)R\mathbf{e}_R \otimes \nabla f_k + kR\nabla f_k \otimes \mathbf{e}_R + R^2\nabla \otimes \nabla f_k = \widehat{\mathbf{F}}_k - k\widehat{\mathbf{T}}_k,\\
&-k(k+1)\mathbf{e}_R \otimes \mathbf{e}_R f_k + (k+2)R\mathbf{e}_R \otimes \nabla f_k - (k+1)R\nabla f_k \otimes \mathbf{e}_R + R^2\nabla \otimes \nabla f_k = \widehat{\mathbf{G}}_k + (k+1)\widehat{\mathbf{T}}_k,\\
&(k+1)(k+3)\mathbf{e}_R \otimes \mathbf{e}_R f_k - (k+1)R\mathbf{e}_R \otimes \nabla f_k - (k+1)R\nabla f_k \otimes \mathbf{e}_R + R^2\nabla \otimes \nabla f_k = \widehat{\mathbf{H}}_{k+2} + (k+1)\widehat{\mathbf{T}}_k.
\end{aligned} \tag{66}$$

This is a system of four linear algebraic equations with four unknowns, namely,

$$\mathbf{e}_R \otimes \mathbf{e}_R f_k, \quad R\mathbf{e}_R \otimes \nabla f_k, \quad R\nabla f_k \otimes \mathbf{e}_R, \quad R^2\nabla \otimes \nabla f_k.$$

Thus, each of these unknowns can be expressed in terms of the four dyadic surface spherical harmonics (65). However, since for our problem, we are interested in $\mathbf{e}_R \otimes \mathbf{e}_R f_k$ only, we only list the solution for it:

$$\mathbf{e}_R \otimes \mathbf{e}_R f_k = \frac{1}{2k+1}\left[\frac{1}{2k+3}(\widehat{\mathbf{H}}_{k+2} - \widehat{\mathbf{G}}_k) - \frac{1}{2k-1}(\widehat{\mathbf{F}}_k - \widehat{\mathbf{E}}_{k-2})\right]. \tag{67}$$

In using Eq. (67), it should be assumed that $\widehat{\mathbf{E}}_{-1} = \hat{\mathbf{0}}$, $\widehat{\mathbf{E}}_{-2} = \hat{\mathbf{0}}$ where $\hat{\mathbf{0}}$ designates the null dyad.

In consequence of the relation (67), Eq. (61) becomes

$$\mathbf{u}^{(1)} = \frac{3}{5-6\nu}\sum_{k=0}^{\infty}\left[2(1-\nu)\widehat{\mathbf{T}}_k - \frac{1}{2k+1}\left\{\frac{1}{2k+3}(\widehat{\mathbf{H}}_{k+2} - \widehat{\mathbf{G}}_k) - \frac{1}{2k-1}(\widehat{\mathbf{F}}_k - \widehat{\mathbf{E}}_{k-2})\right\}\right] \bullet \mathbf{u}_0 \tag{68}$$

on $\partial\Omega_0$.

Thus, expansion of the boundary condition into surface spherical harmonics for the first order perturbation problem is accomplished and the solution for the first order perturbation field is straightforwardly obtained using the general solution (28) as

$$\begin{aligned}
\mathbf{u}^{(1)} = \frac{3}{5-6\nu}\sum_{k=0}^{\infty}\Bigg[&\left(\frac{a}{R}\right)^{k+1}\left\{2(1-\nu)\widehat{\mathbf{T}}_k + \frac{\widehat{\mathbf{G}}_k}{(2k+1)(2k+3)} + \frac{\widehat{\mathbf{F}}_k}{4k^2-1}\right\}_0\\
&-\left(\frac{a}{R}\right)^{k+3}\frac{\widehat{\mathbf{H}}_{k+2}}{(2k+1)(2k+3)} - \left(\frac{a}{R}\right)^{k-1}\frac{\widehat{\mathbf{E}}_{k-2}}{4k^2-1} + \frac{R^2-a^2}{2(3k+5-2\nu(2k+3))}\nabla\nabla\\
&\bullet\left\{\left(\frac{a}{R}\right)^{k+1}\left(2(1-\nu)\widehat{\mathbf{T}}_k + \frac{\widehat{\mathbf{G}}_k}{(2k+1)(2k+3)} + \frac{\widehat{\mathbf{F}}_k}{4k^2-1}\right)\right\}\\
&-\frac{R^2-a^2}{2(3k+11-2\nu(2k+7))(2k+1)(2k+3)}\nabla\nabla\bullet\left\{\left(\frac{a}{R}\right)^{k+1}\widehat{\mathbf{H}}_{k+2}\right\}\\
&-\frac{R^2-a^2}{2(3k-1-2\nu(2k-1))(4k^2-1)}\nabla\nabla\bullet\left\{\left(\frac{a}{R}\right)^{k-1}\widehat{\mathbf{E}}_{k-2}\right\}\Bigg] \bullet \mathbf{u}_0
\end{aligned} \tag{69}$$

in Ω_0. Thus, the solution of the problem up to the first order in ε is given by $\mathbf{u} = \mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)}$, where $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ are given by Eqs. (55) and (69).

7. The net force and the net torque exerted by the inclusion

In this section, we derive expressions for the net force and torque exerted on the medium by the inclusion. They are given by Eqs. (12) from which it is evident that the strain tensor corresponding to the displacement fields (50) and (69) need to be calculated and this could be quite a laborious job. In what follows we will proceed along a different route based on Betti's reciprocal theorem. Once again, we will treat the cases of translation and rotation separately.

7.1. Translation

Consider the following two elastic states. The first state corresponds to the infinite elastic medium with the reference spherical inclusion. This reference inclusion is given an arbitrary translation $\bar{\mathbf{u}}_0$ with respect to its center. The second state corresponds to the nominally spherical inclusion subjected to a translation $\mathbf{u}_0$ with regard to the center of the reference sphere. Thus the boundary conditions for these states can be written as

$$^{(1)}\mathbf{u} = \bar{\mathbf{u}}_0 \quad \text{on } \partial\Omega_0, \qquad ^{(2)}\mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega.$$

Reader would notice that for Betti's reciprocal theorem to be applicable, it is necessary that the bodies (their configurations, boundaries, etc.) in both states have to be the same. Thus, apparently, Betti's theorem can not be applied to these states since the first state involves the reference spherical inclusion while the second the nominally spherical inclusion. However, note that since $\partial\Omega$ differs slightly from $\partial\Omega_0$, therefore the boundary condition corresponding to the second state can be transferred onto $\partial\Omega_0$ via the relations (19). It is on this proviso that we can apply Betti's reciprocal theorem to these states. Furthermore, in applying Betti's reciprocal theorem to bodies whose boundaries extend to infinity, the properties of the source and the field at infinity need to be accommodated. This observation is due to Gurtin and Sternberg (1961) (see also Gurtin, 1972). Following these authors, we consider the outer boundaries of the bodies to be bounded by a spherical shell of a large radius ρ ($\rho \gg a$). With these assumptions, Betti's reciprocal theorem applied to the above states reads as

$$\int_{\partial\Omega_0} \bar{\mathbf{u}}_0 \bullet (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \int_{\partial\Omega_\rho} \bar{\mathbf{u}}_0 \bullet (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\tau$$
$$= \int_{\partial\Omega_0} ({}^{(2)}\mathbf{u}^{(0)} + \varepsilon {}^{(2)}\mathbf{u}^{(1)}) \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \int_{\partial\Omega_\rho} ({}^{(2)}\mathbf{u}^{(0)} + \varepsilon {}^{(2)}\mathbf{u}^{(1)}) \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \mathrm{O}(\varepsilon^2). \tag{70}$$

Now, let $\rho \to \infty$ and note that the displacement vector attenuates at most as $\mathrm{O}(\rho^{-1})$ when $\rho \to \infty$, and so the stress vector corresponding to the spherical surface $\partial\Omega_\rho$ decays as $\mathrm{O}(\rho^{-2})$ when $\rho \to \infty$. Therefore, the contributions from $\partial\Omega_\rho$ as $\rho \to \infty$ can be ignored. Thus, (70) can be rewritten as

$$\int_{\partial\Omega_0} \bar{\mathbf{u}}_0 \bullet (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = \int_{\partial\Omega_0} ({}^{(2)}\mathbf{u}^{(0)} + \varepsilon {}^{(2)}\mathbf{u}^{(1)}) \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \mathrm{O}(\varepsilon^2). \tag{71}$$

Now, since $\bar{\mathbf{u}}_0$ is a constant vector, we can write the integral on the left hand side of Eq. (71) as

$$\bar{\mathbf{u}}_0 \bullet \int_{\partial\Omega_0} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\,\mathrm{d}\tau. \tag{72}$$

Note that since the origin is a singular point for ${}^{(2)}\hat{\boldsymbol{\sigma}}$, we can write

$$\int_{\partial\Omega_0} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\,\mathrm{d}\tau = \int_{\partial\Omega_d} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\,\mathrm{d}\tau + \int_{\partial\Omega_d \cup \partial\Omega_0} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\,\mathrm{d}\tau, \tag{73}$$

where $\partial\Omega_d$ is the surface of a sphere of radius d where $0 < d \ll a$ with the center at the origin. Now, since the integrand in the second integral in (73) does not have any singularity, we can use Gauss's divergence theorem to convert it to a volume integral enclosed by the spherical boundaries $\partial\Omega_d$ and $\partial\Omega_0$:

$$\int_{\partial\Omega_d \cup \partial\Omega_0} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\tau = \int_{\Omega'} \nabla \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\Omega, \tag{74}$$

where Ω' is the volume enclosed by the boundaries $\partial\Omega_d, \partial\Omega_0$. Now, since the medium is in equilibrium, $\nabla \bullet {}^{(2)}\hat{\boldsymbol{\sigma}} = 0$ in Ω'.[3] Thus, the second integral in (73) is zero, and Eq. (74) reduces to

$$\int_{\partial\Omega_0} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\tau = \int_{\partial\Omega_d} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\tau. \tag{75}$$

From Eq. (75), it follows that the surface integral in (72) is independent of the bounding surface and hence it can be taken over any closed surface including the origin. We choose to take $\partial\Omega$, the bounding surface of the nominally spherical inclusion, as the bounding surface. Thus,

$$\int_{\partial\Omega_0} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\tau = \int_{\partial\Omega} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\tau. \tag{76}$$

But, the integral on the right hand side of the equation in (76) is precisely the net force exerted by the inclusion on the medium, i.e.

$$\mathbf{P} = \int_{\partial\Omega} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, \mathrm{d}\tau. \tag{77}$$

In view of (77), the integral on the left hand side of Eq. (71) assumes the form:

$$\int_{\partial\Omega_0} \bar{\mathbf{u}}_0 \bullet (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\, \mathrm{d}\Omega = \bar{\mathbf{u}}_0 \bullet \mathbf{P}. \tag{78}$$

Now, let us turn to evaluate the integral on the right hand side of Eq. (71). Obviously, we have ${}^{(2)}\mathbf{u}^{(0)}|_{\partial\Omega_0} = \mathbf{u}_0$ and ${}^{(2)}\mathbf{u}^{(1)}|_{\partial\Omega_0}$ is given by (61), while $\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}}|_{\partial\Omega_0}$ by (56). In consideration of these relations, it follows that

$$\int_{\partial\Omega_0} ({}^{(2)}\mathbf{u}^{(0)} + \varepsilon\, {}^{(2)}\mathbf{u}^{(1)}) \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\, \mathrm{d}\tau = \mathbf{u}_0 \bullet \mathbf{P}^{(0)} + \varepsilon \frac{6\mu(1-\nu)}{(5-6\nu)a} \mathbf{u}_0 \bullet \int_{\partial\Omega_0} \mathbf{u}^{(1)}\, \mathrm{d}\tau. \tag{79}$$

In view of (68), the integral on the right hand side of Eq. (79) takes the form:

$$\int_{\partial\Omega_0} \mathbf{u}^{(1)}\, \mathrm{d}\tau = \frac{3}{5-6\nu} \int_{\partial\Omega_0} \left[2(1-\nu)\widehat{\mathbf{T}}_0 - \widehat{\mathbf{F}}_0 + \frac{\widehat{\mathbf{G}}_0}{3} - \frac{\widehat{\mathbf{E}}_0}{15} \right] \mathrm{d}\tau. \tag{80}$$

Notice that in view of the orthogonality property of surface spherical harmonics of different degrees over the surface of a unit sphere, only 0th order harmonics are retained in the integrand in Eq. (80).

However, from (65) we have that

$$\widehat{\mathbf{F}}_0 = 0, \quad \widehat{\mathbf{G}}_0 = -\widehat{\mathbf{T}}_0 = -\widehat{\mathbf{I}} f_0, \quad \widehat{\mathbf{E}}_0 = \nabla \otimes \nabla (R^2 f_2). \tag{81}$$

[3] Strictly speaking, the volume Ω' might be comprised of 'chunks' of elastic regions as well as absolutely rigid regions. Equation $\nabla \bullet {}^{(2)}\hat{\boldsymbol{\sigma}} = 0$ is valid for absolutely rigid regions as well, since the stress tensor ${}^{(2)}\hat{\boldsymbol{\sigma}}$ is identically zero in such regions.

Therefore Eq. (80) takes the form:

$$\int_{\partial\Omega_0} \mathbf{u}^{(1)}\,\mathrm{d}\tau = 4\pi a^2\left[\widehat{\mathbf{I}}f_0 - \frac{1}{5(5-6v)}\nabla\otimes\nabla(R^2 f_2)\right]\bullet\bar{\mathbf{u}}_0. \tag{82}$$

In consequence of the relations (78) and (82), Eq. (79) delivers to

$$\bar{\mathbf{u}}_0\bullet\mathbf{P} = \mathbf{P}^{(0)}\bullet\mathbf{u}_0 + \varepsilon\mathbf{u}_0\bullet\widehat{\boldsymbol{\Gamma}}^{(1)}\bullet\bar{\mathbf{u}}_0 + \mathrm{O}(\varepsilon^2), \tag{83}$$

where

$$\widehat{\boldsymbol{\Gamma}}^{(1)} = \frac{24\mu\pi(1-v)a}{5-6v}\left[\widehat{\mathbf{I}}f_0 - \frac{1}{5(5-6v)}\nabla\otimes\nabla(R^2 f_2)\right]. \tag{84}$$

If we denote by $\widehat{\boldsymbol{\Gamma}}$ the stiffness tensor corresponding to the nominally spherical inclusion, then, by Eqs. (59) and (60), it follows that the force $\mathbf{P}$ exerted by it on the medium due to the translation $\mathbf{u}_0$ (with regard to the center of the perfect sphere) is

$$\mathbf{P} = \widehat{\boldsymbol{\Gamma}}\bullet\mathbf{u}_0. \tag{85}$$

On the other hand, the force $\mathbf{P}^{(0)}$ exerted by the reference spherical inclusion on the medium due to the translation $\bar{\mathbf{u}}_0$ is given by

$$\mathbf{P}^{(0)} = \widehat{\boldsymbol{\Gamma}}^{(0)}\bullet\bar{\mathbf{u}}_0. \tag{86}$$

Putting Eqs. (85) and (86) into (83), we find that

$$\bar{\mathbf{u}}_0\bullet\widehat{\boldsymbol{\Gamma}}\bullet\mathbf{u}_0 = \mathbf{u}_0\bullet(\widehat{\boldsymbol{\Gamma}}^{(0)} + \varepsilon\widehat{\boldsymbol{\Gamma}}^{(1)})\bullet\bar{\mathbf{u}}_0 + \mathrm{O}(\varepsilon^2). \tag{87}$$

From Eq. (87), it follows that up to $\mathrm{O}(\varepsilon)$, the stiffness tensor for the medium with the nominally spherical inclusion is given by

$$\widehat{\boldsymbol{\Gamma}} = \widehat{\boldsymbol{\Gamma}}^{(0)} + \varepsilon\widehat{\boldsymbol{\Gamma}}^{(1)} + \mathrm{O}(\varepsilon^2). \tag{88}$$

Thus, up to $\mathrm{O}(\varepsilon)$, the net force exerted by the nominally spherical inclusion to due a constant translation $\mathbf{u}_0$ with respect to the center of the reference spherical inclusion is

$$\mathbf{P} = \widehat{\boldsymbol{\Gamma}}\bullet\mathbf{u}_0 = \frac{24\pi\mu(1-v)a}{5-6v}\left[(1+\varepsilon f_0)\mathbf{u}_0 - \varepsilon\frac{1}{5(5-6v)}\nabla\otimes\nabla(R^2 f_2)\bullet\mathbf{u}_0\right]. \tag{89}$$

Eq. (89) shows that in general the direction of the net force required to produce the constant translation of a nominally spherical inclusion is different from that of the applied translation.

In addition, the nominally spherical inclusion would exert on the medium a torque about the origin, which can be calculated using the equation:

$$\mathbf{M} = \int_{\partial\Omega} \mathbf{R}\times(\mathbf{n}\bullet\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau. \tag{90}$$

To evaluate (90), we again make use of Betti's reciprocal theorem. For the first state, we take the reference spherical inclusion subjected to a small arbitrary rotation $\boldsymbol{\omega}$ around its center, while for the second state we take the case of the nominally spherical inclusion subjected to the translation $\mathbf{u}_0$ with respect to the center of the reference spherical inclusion. Using the same arguments as before, we deduce that

$$\int_{\partial\Omega_0}(\boldsymbol{\omega}\times\mathbf{R})\bullet(\mathbf{n}\bullet{}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = \int_{\partial\Omega_0}(\mathbf{u}_0 + \varepsilon\mathbf{u}^{(1)})\bullet(\mathbf{n}\bullet{}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \mathrm{O}(\varepsilon^2). \tag{91}$$

Considering the identity

$$(\boldsymbol{\omega} \times \mathbf{R}) \bullet (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}) = (\mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})) \bullet \boldsymbol{\omega},$$

we rewrite (91) as

$$\boldsymbol{\omega} \bullet \int_{\partial\Omega_0} (\mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}))\,\mathrm{d}\tau = \int_{\partial\Omega_0} (\mathbf{u}_0 + \varepsilon\mathbf{u}^{(1)}) \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \mathrm{O}(\varepsilon^2). \tag{92}$$

As before, note that since the origin is a singular point for ${}^{(2)}\hat{\boldsymbol{\sigma}}$, we can write the surface integral on the left hand side of Eq. (92) as

$$\int_{\partial\Omega_0} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = \int_{\partial\Omega_d} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau + \int_{\partial\Omega_d \cup \partial\Omega_0} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau, \tag{93}$$

where $\partial\Omega_d$ is the surface of a sphere of radius d where $0 < d \ll a$ with the center at the origin. Now, the integrand in the second surface integral on the right hand side of Eq. (93) does not have any singularity and it can be converted, by means of some well-known results from tensor analysis (see, for instance, Lure, 1970, p. 847) to the following volume integral enclosed by the spherical boundaries $\partial\Omega_d$ and $\partial\Omega_0$:

$$\int_{\partial\Omega_d \cup \partial\Omega_0} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = \int_{\Omega'} (\mathbf{R} \times \nabla \bullet {}^{(2)}\hat{\boldsymbol{\sigma}} - 2\boldsymbol{\gamma})\,\mathrm{d}\Omega, \tag{94}$$

where

$$\boldsymbol{\gamma} = \frac{-1}{2}\mathbf{e}_m \times (\mathbf{e}_m \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}). \tag{95}$$

Since the stress-tensor in linearized elasticity theory is symmetric, it is a simple exercise to show from (95) that $\boldsymbol{\gamma} = 0$. Furthermore, since the medium is in equilibrium, $\nabla \bullet {}^{(2)}\hat{\boldsymbol{\sigma}} = 0$ in Ω'. Thus, the integral in (94) is zero and Eq. (93) assumes the form:

$$\int_{\partial\Omega_0} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = \int_{\partial\Omega_d} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau. \tag{96}$$

Therefore, the surface integral on the left hand side of Eq. (96) is independent of the bounding surface and hence it can be taken over any closed surface around the origin. As before, we choose to take the surface of the nominally spherical inclusion, $\partial\Omega$, as the bounding surface. Thus,

$$\int_{\partial\Omega_0} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = \int_{\partial\Omega} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau. \tag{97}$$

But, the integral on the right hand side of the equation in (97) is precisely the net torque exerted by the nominally spherical inclusion on the medium, i.e.

$$\mathbf{M} = \int_{\partial\Omega} \mathbf{R} \times (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau. \tag{98}$$

Therefore, Eq. (92) assumes the form:

$$\boldsymbol{\omega} \bullet \mathbf{M} = \int_{\partial\Omega_0} (\mathbf{u}_0 + \varepsilon\mathbf{u}^{(1)}) \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\Omega_0 + \mathrm{O}(\varepsilon^2). \tag{99}$$

Proceeding now on to evaluate the surface integral on the right hand side of Eq. (99), we note that $\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}}$ is given by Eq. (56). Thus,

$$\begin{aligned}\int_{\partial\Omega_0} \mathbf{u}_0 \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau &= \mathbf{u}_0 \bullet \int_{\partial\Omega_0} (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\,\mathrm{d}\tau = -3\mu\mathbf{u}_0 \bullet \int_{\partial\Omega_0} \mathbf{e}_R \times \boldsymbol{\omega}\,\mathrm{d}\tau \\ &= -3\mu\mathbf{u}_0 \bullet \left\{ \left(\int_{\partial\Omega_0} \mathbf{e}_R\,\mathrm{d}\tau \right) \times \boldsymbol{\omega} \right\} = 0, \quad \text{since } \int_{\partial\Omega_0} \mathbf{e}_R\,\mathrm{d}\tau = 0.\end{aligned} \tag{100}$$

Hence, Eq. (99) delivers to the form:

$$\boldsymbol{\omega} \bullet \mathbf{M} = -\varepsilon 3\mu \int_{\partial\Omega_0} \mathbf{u}^{(1)} \bullet (\mathbf{e}_R \times \boldsymbol{\omega})\, d\tau + O(\varepsilon^2). \tag{101}$$

However, owing to the identity $\mathbf{u}^{(1)} \bullet (\mathbf{e}_R \times \boldsymbol{\omega}) = (\mathbf{u}^{(1)} \times \mathbf{e}_R) \bullet \boldsymbol{\omega}$, Eq. (101) reduces to

$$\boldsymbol{\omega} \bullet \mathbf{M} = -\varepsilon 3\mu\boldsymbol{\omega} \bullet \int_{\partial\Omega_0} (\mathbf{u}^{(1)} \times \mathbf{e}_R)\, d\tau + O(\varepsilon^2), \tag{102}$$

whence it follows that

$$\mathbf{M} = -\varepsilon 3\mu \int_{\partial\Omega_0} \mathbf{u}^{(1)} \times \mathbf{e}_R\, d\tau + O(\varepsilon^2). \tag{103}$$

Putting the expression for $\mathbf{u}^{(1)}$ from (61) into (103), we obtain

$$\mathbf{M} = \varepsilon \frac{-9}{5-6v} \sum_{k=0}^{\infty} [2(1-v)\hat{\boldsymbol{\alpha}}_1 - \hat{\boldsymbol{\alpha}}_2] \bullet \mathbf{u}_0 + O(\varepsilon^2), \tag{104}$$

where

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_1 &= \int_{\partial\Omega_0} \widehat{\mathbf{T}}_k \times \mathbf{e}_R\, d\tau, \\ \hat{\boldsymbol{\alpha}}_2 &= \int_{\partial\Omega_0} (\mathbf{e}_R \otimes \mathbf{e}_R) \times \mathbf{e}_R f_k\, d\tau. \end{aligned} \tag{105}$$

Considering the first integral in (105), we can write

$$\hat{\boldsymbol{\alpha}}_1 = \int_{\partial\Omega_0} \widehat{\mathbf{I}} \times \mathbf{e}_R f_k\, d\tau = \widehat{\mathbf{I}} \times \left(\int_{\partial\Omega_0} \mathbf{e}_R f_k\, d\tau\right). \tag{106}$$

Substituting the expression for $\mathbf{e}_R f_k$ from (49) into (106), we obtain

$$\hat{\boldsymbol{\alpha}}_1 = \frac{1}{2k+1} \widehat{\mathbf{I}} \times \left(\int_{\partial\Omega_0} (\mathbf{C}_{k-1} - \mathbf{D}_{k+1})\, d\tau\right) = \frac{4\pi a^2 \delta_{k1}}{2k+1} \widehat{\mathbf{I}} \times \mathbf{C}_0, \tag{107}$$

where δ_{ij} is the Kronecker's delta.

Now take the second integral in (105); we observe that

$$\hat{\boldsymbol{\alpha}}_2 = \int_{\partial\Omega_0} (\mathbf{e}_R \otimes \mathbf{e}_R) \times \mathbf{e}_R f_k\, d\tau = \int_{\partial\Omega_0} (\mathbf{e}_R \times \mathbf{e}_R) \otimes \mathbf{e}_R f_k\, d\tau = \hat{\mathbf{0}}. \tag{108}$$

Thus, in view of relations (107) and (108), Eq. (104) delivers to the following final form:

$$\mathbf{M} = \varepsilon a \nabla(R f_1) \times \frac{24\pi\mu(1-v)a}{5-6v} \mathbf{u}_0 + O(\varepsilon^2). \tag{109}$$

Note that the force exerted by the nominally spherical inclusion on the medium is given by Eq. (89). Thus, up to $O(\varepsilon)$ Eq. (109) can be represented as

$$\mathbf{M} = \varepsilon a \nabla(R f_1) \times \int_{\partial\Omega} \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, d\Omega + O(\varepsilon^2).$$

Subtracting the above equation from (90), we see that up to the first order in ε

$$\int_{\partial\Omega} [\mathbf{R} - \varepsilon a \nabla(R f_1)] \times \mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}}\, d\Omega = 0. \tag{110}$$

Thus, it follows from (110) that if the inclusion is given the translation with regard to the point $\mathbf{R}_0 = \varepsilon a \nabla(Rf_1)$, the inclusion would not exert on the medium any torque. Following Brenner (1964, 1963), we call this point the center of elastostatic stresses. As shown by Brenner (1964), up to the first order in ε, this point corresponds to the centroid of the nominally spherical inclusion (see Appendix A).

We now proceed to the case of rotation.

7.2. Rotation

The toque exerted by the inclusion on the medium is given by

$$\mathbf{M} = \int_{\partial\Omega} \mathbf{R} \times (\mathbf{n} \bullet \hat{\boldsymbol{\sigma}})\, \mathrm{d}\tau. \tag{111}$$

Here $\hat{\boldsymbol{\sigma}}$ is the stress-tensor corresponding to the case of rotation of the nominally spherical inclusion. To evaluate (111), we again use Betti's reciprocal theorem. Consider two elastic states in which the first state corresponds to the perfectly spherical inclusion subjected to the rotation $\bar{\boldsymbol{\omega}}$ around its center while the second state to the nominally spherical inclusion subjected to the rotation $\boldsymbol{\omega}$ around the center of the reference spherical inclusion. Applying Betti's reciprocal theorem to these states and following the same lines of arguments as those used for the case of translation, we obtain

$$\int_{\partial\Omega_0} (\bar{\boldsymbol{\omega}} \times \mathbf{R}) \bullet (\mathbf{n} \bullet {}^{(2)}\hat{\boldsymbol{\sigma}})\, \mathrm{d}\tau = \int_{\partial\Omega_0} {}^{(2)}\mathbf{u} \bullet (\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}})\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2). \tag{112}$$

In Eq. (112), $\mathbf{n} \bullet {}^{(1)}\hat{\boldsymbol{\sigma}}$ is given by (34) with $\boldsymbol{\omega}$ replaced by $\bar{\boldsymbol{\omega}}$, and ${}^{(2)}\mathbf{u}$ by (50). In consideration of these equations and using arguments similar to those used for the case of pure translation, we deduce that

$$\bar{\boldsymbol{\omega}} \bullet \mathbf{M} = \boldsymbol{\omega} \bullet \overline{\mathbf{M}}^{(0)} - \varepsilon 3\mu \int_{\partial\Omega_0} \mathbf{u}_1 \bullet (\mathbf{e}_R \times \bar{\boldsymbol{\omega}})\, \mathrm{d}\Omega_0. \tag{113}$$

Since $\mathbf{u}_1 \bullet (\mathbf{e}_R \times \bar{\boldsymbol{\omega}}) = \bar{\boldsymbol{\omega}} \bullet (\mathbf{u}_1 \times \mathbf{e}_R)$, Eq. (113) then reduces to

$$\bar{\boldsymbol{\omega}} \bullet \mathbf{M} = \boldsymbol{\omega} \bullet \overline{\mathbf{M}}^{(0)} - \varepsilon 3\mu \bar{\boldsymbol{\omega}} \bullet \int_{\partial\Omega_0} (\mathbf{u}_1 \times \mathbf{e}_R)\, \mathrm{d}\tau. \tag{114}$$

Next, putting the expression for $\mathbf{u}_1$ from (50) into (114), after some simple vector operations, we obtain

$$\bar{\boldsymbol{\omega}} \bullet \mathbf{M} = \boldsymbol{\omega} \bullet \overline{\mathbf{M}}^{(0)} + \varepsilon 9\mu a \bar{\boldsymbol{\omega}} \bullet \widehat{\boldsymbol{\Lambda}}^{(1)} \bullet \boldsymbol{\omega}, \tag{115}$$

where

$$\widehat{\boldsymbol{\Lambda}}^{(1)} = \sum_{k=0}^{\infty} \int_{\partial\Omega_0} (\widehat{\mathbf{I}} f_k - \mathbf{e}_R \otimes \mathbf{e}_R f_k)\, \mathrm{d}\tau. \tag{116}$$

To evaluate (116), we make use of the relations (67) and (81), and note that owing to the orthogonality property of surface spherical harmonics of different degrees over the surface of a unit sphere, only zeroth degree surface spherical harmonics should be retained; we thus obtain

$$\widehat{\boldsymbol{\Lambda}}^{(1)} = \frac{4\pi a^2}{15} [10 \widehat{\mathbf{I}} f_0 - \nabla \otimes \nabla (R^2 f_2)]. \tag{117}$$

Next, since $\mathbf{M}$ is the net torque corresponding to the nominally spherical inclusion subjected to the rotation $\boldsymbol{\omega}$ and $\overline{\mathbf{M}}^{(0)}$ is that corresponding to the perfectly spherical inclusion subjected to the rotation $\bar{\boldsymbol{\omega}}$, they can be represented as

$$\mathbf{M} = \widehat{\boldsymbol{\Lambda}} \bullet \boldsymbol{\omega}, \quad \overline{\mathbf{M}}^{(0)} = \widehat{\boldsymbol{\Lambda}}^{(0)} \bullet \bar{\boldsymbol{\omega}}. \tag{118}$$

Putting (118) into (115), we obtain

$$\bar{\boldsymbol{\omega}} \bullet \widehat{\boldsymbol{\Lambda}} \bullet \boldsymbol{\omega} = \boldsymbol{\omega} \bullet (\widehat{\boldsymbol{\Lambda}}^{(0)} + \varepsilon 9\mu a \widehat{\boldsymbol{\Lambda}}^{(1)}) \bullet \bar{\boldsymbol{\omega}}. \tag{119}$$

It therefore follows from (119) that

$$\widehat{\boldsymbol{\Lambda}} = \widehat{\boldsymbol{\Lambda}}^{(0)} + \varepsilon 9\mu a \widehat{\boldsymbol{\Lambda}}^{(1)}. \tag{120}$$

Thus, the rotational stiffness tensor for the nominally spherical inclusion is given by Eq. (120) and this tensor is a symmetric one.

Thus, up to the first order in ε, the torque exerted by the nominally spherical inclusion subjected to a rotation $\boldsymbol{\omega}$ around the center of the reference spherical inclusion is given by

$$\mathbf{M} = \widehat{\boldsymbol{\Lambda}} \bullet \boldsymbol{\omega} = \mathbf{M}^{(0)} + \varepsilon \mathbf{M}^{(1)} = 8\pi\mu a^3 \left[(1 + 3\varepsilon f_0)\boldsymbol{\omega} - \varepsilon \frac{3}{10} \nabla \otimes \nabla (R^2 f_2) \bullet \boldsymbol{\omega} \right]. \tag{121}$$

In addition, the inclusion would exert a force on the medium. This force is calculated using the equation

$$\mathbf{P} = \int_{\partial\Omega} \mathbf{n} \bullet \hat{\boldsymbol{\sigma}} \, d\tau.$$

Again, we use Betti's reciprocal theorem. For the two elastic states, we take the perfectly spherical inclusion subjected to a constant translation $\mathbf{u}_0$. For the second state we take the nominally spherical inclusion subjected to the rotation $\boldsymbol{\omega}$. Then following the same lines of arguments as in the case of translation, we deduce that up to the first order in ε, the force exerted by the inclusion on the medium is given by

$$\mathbf{P} = \varepsilon \frac{6\mu(1-v)}{(5-6v)a} \int_{\partial\Omega_0} \mathbf{u}^{(1)} \, d\tau + O(\varepsilon^2), \tag{122}$$

where $\mathbf{u}^{(1)}$ is given by (61), which, upon substitution into (122), yields the relation

$$\begin{aligned}\mathbf{P} &= \varepsilon \frac{6\pi\mu(1-v)}{(5-6v)} \boldsymbol{\omega} \times \int_{\partial\Omega_0} \mathbf{C}_0 \, d\tau + O(\varepsilon^2) = \varepsilon \frac{24\pi\mu(1-v)a^2}{(5-6v)} \boldsymbol{\omega} \times \mathbf{C}_0 + O(\varepsilon^2) \\ &= \varepsilon \frac{24\pi\mu(1-v)a^2}{(5-6v)} \boldsymbol{\omega} \times \nabla(R f_1) + O(\varepsilon^2).\end{aligned} \tag{123}$$

It is well known that in the special where the medium is an incompressible one (with Poisson's ration $v = 1/2$) and if the displacement vector is interpreted as the velocity vector, equations of elastostatics reduce to those for slow steady viscous flow of a fluid also known as Stokes flow. It can be seen from Eqs. (89), (109), (121) and (123) that they are consistent with Brenner's results for the low Reynolds number resistance of a slightly deformed spherical particle to small translational and rotational motions (see Brenner, 1964; Happel and Brenner, 1973).

We now consider some concrete examples of boundary perturbations:

Example 1. Consider first a rather trivial case where the shape perturbation is given by $f = -1$. Thus, in this case $f_0 = -1$, all other members of the sequence $\{f_k\}$ being zero.

Consider first the case of pure translation. With reference to Eq. (89), the net force exerted by the inclusion on the medium is given by

$$\mathbf{P} = \frac{24\mu\pi(1-v)a(1-\varepsilon)}{5-6v} \mathbf{u}_0. \tag{124}$$

Note that as a result of this shape perturbation, the original spherical inclusion of radius a again becomes a perfectly spherical inclusion of radius $a(1-\varepsilon)$. Therefore, the exact net force, as per Eq. (57), is

$$\mathbf{P} = \frac{24\mu\pi(1-v)a(1-\varepsilon)}{5-6v}\mathbf{u}_0,$$

which agrees with Eq. (124). The inclusion does not exert any torque on the medium.

For the case of pure rotation, the net torque exerted by the inclusion on the medium, as per Eq. (121), is given by

$$\mathbf{M} = 8\pi\mu a^3(1-3\varepsilon)\boldsymbol{\omega} + \mathrm{O}(\varepsilon^2). \tag{125}$$

The exact net torque, as per Eq. (38), is

$$\mathbf{M} = 8\pi\mu a^3(1-\varepsilon)^3\boldsymbol{\omega},$$

which, to $\mathrm{O}(\varepsilon)$ agrees with Eq. (125).

These simple tests render credence to the correctness of the foregoing analysis.

Example 2. As a second example, consider the case of translation of a prolate spheroid (see Fig. 1)[4] whose equation is given by

$$\frac{x_1^2}{a^2} + \frac{x_2^2+x_3^2}{a^2(1-\varepsilon)^2} = 1, \quad 0 < \varepsilon \ll 1. \tag{126}$$

To $\mathrm{O}(\varepsilon)$, Eq. (126) can be written as

$$R = a\left[1 + \varepsilon\left\{-\frac{2}{3}P_0(\cos\phi) - \frac{1}{3}P_2(\cos\phi) + \frac{1}{6}\cos 2\theta P_2^2(\cos\phi)\right\}\right] + \mathrm{O}(\varepsilon^2). \tag{127}$$

Thus, in the case

$$f_0 = \frac{-2}{3}, \quad f_2 = \frac{-1}{3}P_2(\cos\phi) + \frac{1}{6}\cos 2\theta P_2^2(\cos\phi), \quad R^2 f_2 = \frac{-R^2}{3} + x_1^2.$$

Putting these expressions into (89) and (109), we find that the net force is given by

$$\mathbf{P} = \frac{24\pi\mu(1-v)a}{5-6v}\left[\left\{1 - \varepsilon\frac{2(4-5v)}{5(5-6v)}\right\}\mathbf{u}_0 - \varepsilon\frac{2}{5(5-6v)}\mathbf{e}_1 \otimes \mathbf{e}_1 \bullet \mathbf{u}_0\right] + \mathrm{O}(\varepsilon^2), \tag{128}$$

while the net toque exerted by the inclusion is null.

If we introduce the notation

$$e = \sqrt{1 - \frac{a_2^2}{a_1^2}},$$

then

$$e^2 = 2\varepsilon + \mathrm{O}(\varepsilon^2).$$

Thus, Eq. (128) can be rewritten in terms of the eccentricity, e, of the spheroid as

$$\mathbf{P} = \frac{24\pi\mu(1-v)a}{5-6v}\left[\left\{1 - e^2\frac{4(4-5v)}{5(5-6v)}\right\}\mathbf{u}_0 - e^2\frac{1}{5(5-6v)}\mathbf{e}_1 \otimes \mathbf{e}_1 \bullet \mathbf{u}_0\right] + \mathrm{O}(e^4). \tag{129}$$

[4] For the sake of illustration, the shape perturbations in Figs. 1, 3–6, 8 are grossly exaggerated.

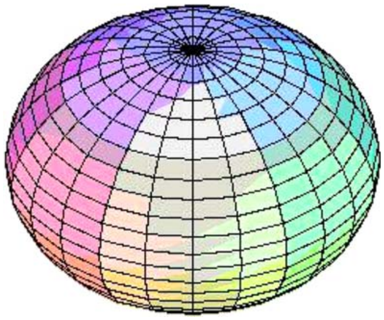

Fig. 1. Prolate spheroidal inclusion (Example 2).

For instance, if the applied translation is in the $x_3 = 0$ plane, then equation $\mathbf{u}_0 \bullet \mathbf{e}_3 = 0$, and Eq. (129) gives

$$\begin{aligned} P_x &= \frac{24\pi\mu(1-v)a}{5-6v}\left[1 - e^2\frac{3(3-4v)}{5(5-6v)}\right]u_{01} + \mathrm{O}(e^4), \\ P_y &= \frac{24\pi\mu(1-v)a}{5-6v}\left[1 - e^2\frac{2(4-5v)}{5(5-6v)}\right]u_{02} + \mathrm{O}(e^4), \\ P_z &= 0. \end{aligned} \tag{130}$$

Kanwal and Sharma (1976) (see also Kanwal, 1983) give the exact solution of this problem using the singularity method. In particular, they show that the components of the net force are

$$\begin{aligned} P_x &= \frac{32\pi\mu(1-v)u_{01}ae^3}{-2e + (1+3e^2-4ve^2)\ln[(1+e)/(1-e)]}, \\ P_y &= \frac{64\pi\mu(1-v)u_{02}ae^3}{2e - (1-7e^2+8ve^2)\ln[(1+e)/(1-e)]}, \\ P_z &= 0. \end{aligned} \tag{131}$$

Fig. 2 illustrates the absolute error of the perturbation solution (130) for different values of the eccentricity of the prolate spheroid (Poisson's ration is assumed to be equal to 0.25). It is clear from this plot that even for a moderately prolate spheroidal inclusion with $e = 0.7$ (i.e. $b = 0.71a$), the absolute error does not exceed 6%. Thus, the first order perturbation solution yields remarkably accurate result. The same type of accuracy may be expected from other boundary perturbations.

Consider now the rotation of the nominally spherical inclusion represented by Eq. (126). Referring to Eqs. (121) and (123), we infer that the inclusion would exert the net torque on the medium

$$\mathbf{M} = 8\pi\mu a^3\left[\boldsymbol{\omega}\left(1 - e^2\frac{9}{10}\right) - e^2\frac{3}{10}\mathbf{e}_1 \otimes \mathbf{e}_1 \bullet \boldsymbol{\omega}\right] + \mathrm{O}(e^4) \tag{132}$$

around the center of the reference sphere while it would not exert any force.

In the special case where the rotation is around the $x_3 = 0$ plane, Eq. (132) simplifies to

$$\mathbf{M} = 8\pi\mu a^3\omega\mathbf{e}_3\left(1 - e^2\frac{9}{10}\right) + \mathrm{O}(e^4).$$

Example 3. As the third example, consider the case where the inclusion has the shape of a cardioid (see Fig. 3), namely,

$$R = a(1 + \varepsilon\cos\phi).$$

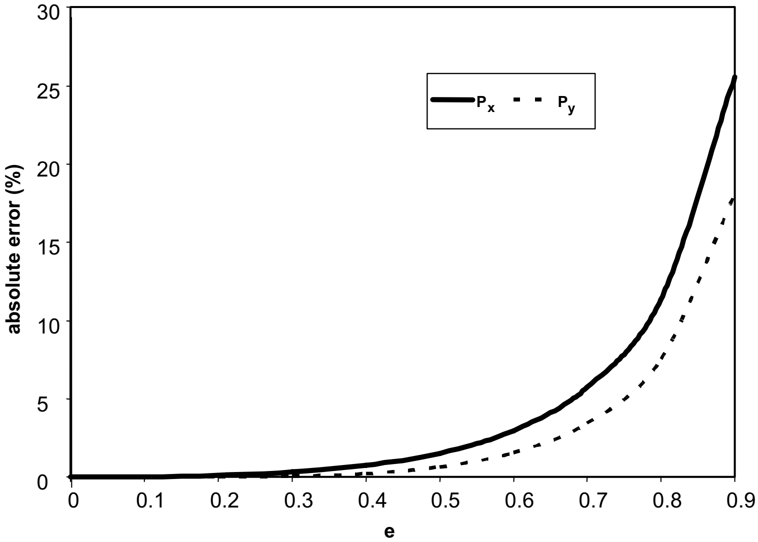

Fig. 2. Absolute error estimates for P_X and P_Y for different values of the eccentricity of the prolate spheroid (Poisson's ratio = 0.25).

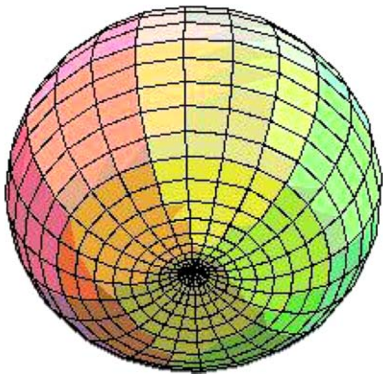

Fig. 3. Cardioid (Example 3).

In this case, the only non-zero member of the sequence $\{f_k\}$ is f_1:

$$f_1 = P_1(\cos\phi). \tag{133}$$

Putting (133) into Eqs. (89) and (109), we see that for the case of pure translation, the net force is zero, but the inclusion would exert a torque with respect to the center of the reference sphere, which is given by

$$\mathbf{M} = \frac{24\pi\mu(1-v)a}{5-6v}\mathbf{u}_0 \times \varepsilon a\mathbf{e}_3 + \mathrm{O}(\varepsilon^2). \tag{134}$$

Note that up to $\mathrm{O}(\varepsilon)$, the centroid of the inclusion is, as per Eq. (A.3), is located at the point $\varepsilon a\mathbf{e}_3$, i.e. on the positive x_3-axis at a distance εa from the origin. Thus, if the translation is given with respect to the point $\varepsilon a\mathbf{e}_3$, then the inclusion would not exert any torque on the medium.

Consider now the case of pure rotation. Putting (133) into Eqs. (121) and (123), we observe that the net toque exerted by the inclusion on the medium is null. However, the inclusion would exert a force which is given by

$$\mathbf{P} = -\varepsilon \frac{24\pi\mu(1-v)a^2}{(5-6v)} \boldsymbol{\omega} \times \mathbf{e}_3 + \mathrm{O}(\varepsilon^2).$$

Example 4. As the fourth example, let us consider the case where the shape perturbation is given by the equation

$$f(\theta, \phi) = \sin\theta\cos\theta\sin^2\phi\cos^2\phi.$$

As a result of this perturbation, the inclusion assumes the shape shown in Fig. 4.

It can be shown using the methods of Section 2 that the above function can be expanded into a series of surface spherical harmonics as

$$\sin\theta\cos\theta\sin^2\phi\cos^2\phi = \frac{1}{42} P_2^2(\cos\phi)\sin 2\theta + \frac{1}{105} P_4^2(\cos\phi)\sin 2\theta.$$

Thus, in the case the non-zero members of the sequence $\{f_k\}$ are

$$f_2 = \frac{1}{42} P_2^2(\cos\phi)\sin 2\theta, \quad f_4 = \frac{1}{105} P_4^2(\cos\phi)\sin 2\theta.$$

Of the two surface spherical harmonics, only f_2 contributes to the first order perturbation field. Thus, if this inclusion is given a translation $\mathbf{u}_0$ with respect to the center of the reference sphere, the inclusion would exert the following force on the medium:

$$\mathbf{P} = \frac{24\pi\mu(1-v)a}{5-6v}\left[\mathbf{u}_0 - \varepsilon \frac{1}{35(5-6v)}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \bullet \mathbf{u}_0\right] + \mathrm{O}(\varepsilon^2). \tag{135}$$

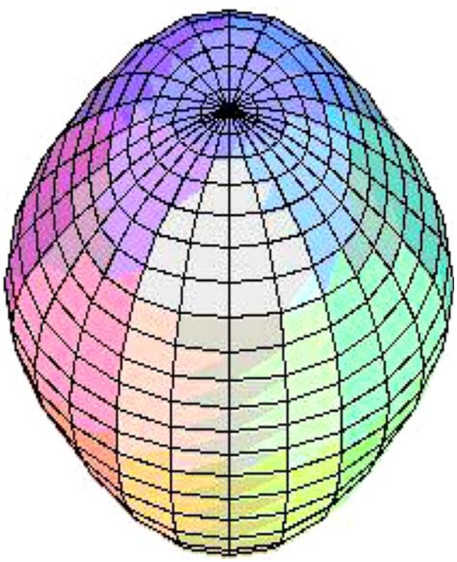

Fig. 4. Inclusion shape corresponding to Example 4.

Eq. (135) can be also written in component form if desired. In particular, if the applied translation is given along the x_1-axis (i.e. $\mathbf{u}_0 = u_0\mathbf{e}_1$), from the formula (135), we find that

$$P_x = \frac{24\pi\mu(1-\nu)au_0}{5-6\nu} + O(\varepsilon^2), \quad P_y = -\varepsilon\frac{24\pi\mu(1-\nu)au_0}{35(5-6\nu)^2} + O(\varepsilon^2), \quad P_z = O(\varepsilon^2).$$

On the other hand, if this inclusion is given a rotation $\boldsymbol{\omega}$ about the center of the reference sphere, the torque necessary to produce this rotation can be obtained by putting the value of f_2 into (121) with the result

$$\mathbf{M} = 8\pi\mu a^3\left[\boldsymbol{\omega} - \varepsilon\frac{3}{70}(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\bullet\boldsymbol{\omega}\right] + O(\varepsilon^2). \tag{136}$$

For instance, if the rotation $\boldsymbol{\omega}$ is given around the x_1-axis (i.e. $\boldsymbol{\omega} = \omega\mathbf{e}_1$), Eq. (136) yields:

$$M_x = 8\pi\mu a^3\omega + O(\varepsilon^2), \quad M_y = -\varepsilon\frac{12\pi\mu a^3\omega}{35} + O(\varepsilon^2), \quad M_z = O(\varepsilon^2).$$

In another special case where the rotation $\boldsymbol{\omega}$ is given about the x_3-axis, Eq. (136) simplifies to

$$M_x = M_y = O(\varepsilon^2), \quad M_z = 8\pi\mu a^3\omega + O(\varepsilon^2).$$

Thus, to the first order in ε, the torque required to produce the rotation $\boldsymbol{\omega} = \omega\mathbf{e}_3$ around the center of the reference sphere is the same as that for the perfectly spherical inclusion.

Example 5. As the next example, we consider the case where the shape perturbation is given by

$$f(\theta,\phi) = \sin\theta\cos^2\theta\sin^3\phi\cos^3\phi.$$

As a result of this perturbation, the inclusion assumes the shape shown in Fig. 5. It can be shown that the function $f(\theta,\phi)$ can be expanded into a series of surface spherical harmonics as

$$f(\theta,\phi) = \left[\frac{1}{6930}P_6^3(\cos\phi) + \frac{1}{1540}P_4^3(\cos\phi)\right]\sin 3\theta - \left[\frac{2}{693}P_6^1(\cos\phi) - \frac{1}{770}P_4^1(\cos\phi) - \frac{1}{63}P_2^1(\cos\phi)\right]\sin\theta.$$

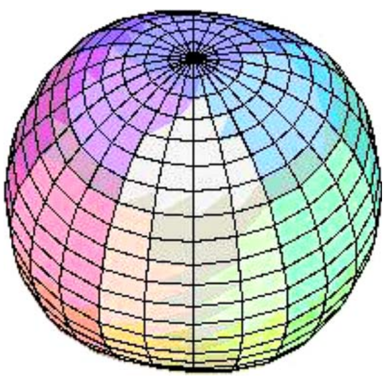

Fig. 5. Inclusion shape corresponding to Example 5.

Thus, in this case

$$f_2 = \frac{1}{63} P_2^1(\cos\phi)\sin\theta, \quad f_4 = \frac{1}{770} P_4^1(\cos\phi)\sin\theta + \frac{1}{1540} P_4^3(\cos\phi)\sin 3\theta,$$
$$f_6 = \frac{1}{6930} P_6^3(\cos\phi)\sin 3\theta - \frac{2}{693} P_6^1(\cos\phi)\sin\theta.$$

Of the above surface spherical harmonics, only f_2 contributes and we have

$$\nabla \otimes \nabla(R^2 f_2) = \frac{1}{21}(\mathbf{e}_3 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_3). \tag{137}$$

Thus, the total force required to produce the translation $\mathbf{u}_0$ of the inclusion with regard to the center of the reference sphere is

$$\mathbf{P} = \frac{24\pi\mu(1-v)a}{5-6v}\left[\mathbf{u}_0 - \varepsilon\frac{1}{105(5-6v)}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \bullet \mathbf{u}_0\right] + \mathrm{O}(\varepsilon^2). \tag{138}$$

For instance, if the applied translation is along the x_2-axis (i.e. $\mathbf{u}_0 = u_0\mathbf{e}_2$), Eq. (138) gives

$$P_x = \mathrm{O}(\varepsilon^2),$$
$$P_y = \frac{24\pi\mu(1-v)au_0}{5-6v} + \mathrm{O}(\varepsilon^2),$$
$$P_z = -\varepsilon\frac{8\pi\mu(1-v)au_0}{35(5-6v)^2} + \mathrm{O}(\varepsilon^2).$$

On the other hand, the net torque required to produce the rotation $\boldsymbol{\omega}$ of this inclusion around the center of the reference sphere is

$$\mathbf{M} = 8\pi\mu a^3\left[\boldsymbol{\omega} - \varepsilon\frac{1}{70}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \bullet \boldsymbol{\omega}\right] + \mathrm{O}(\varepsilon^2). \tag{139}$$

In particular, if the rotation is given around the x_3-axis (i.e. $\boldsymbol{\omega} = \omega\mathbf{e}_3$), Eq. (139) yields

$$M_x = \mathrm{O}(\varepsilon^2), \quad M_y = \varepsilon\frac{-4\pi\mu a^3\omega}{35} + \mathrm{O}(\varepsilon^2), \quad M_z = 8\pi\mu a^3\omega + \mathrm{O}(\varepsilon^2).$$

Example 6. Consider the case where the shape perturbation is specified as

$$f(\theta, \phi) = \cos 2\theta + \cos 2\phi.$$

The resulting inclusion shape is illustrated in Fig. 6. The above shape function can be expanded into a series of surface spherical harmonics. Listing only pertinent to our analysis the first three harmonics, we have

$$f_0 = \frac{-1}{3}, \quad f_1 = 0, \quad f_2 = \frac{5}{4}\cos 2\theta \sin^2\phi + \frac{2}{3}(3\cos^2\phi - 1)$$

so that

$$\nabla \otimes \nabla(R^2 f_2) = \frac{1}{6}(7\mathbf{e}_1 \otimes \mathbf{e}_1 - 23\mathbf{e}_2 \otimes \mathbf{e}_2 + 16\mathbf{e}_3 \otimes \mathbf{e}_3).$$

Thus, if this inclusion is given a translation $\mathbf{u}_0$ with respect to the center of the reference sphere, the inclusion would exert the following force on the medium:

$$\mathbf{P} = \frac{24\pi\mu(1-v)a}{5-6v}\left[\left(1 - \varepsilon\frac{1}{3}\right)\mathbf{u}_0 - \varepsilon\frac{1}{30(5-6v)}(7\mathbf{e}_1 \otimes \mathbf{e}_1 - 23\mathbf{e}_2 \otimes \mathbf{e}_2 + 16\mathbf{e}_3 \otimes \mathbf{e}_3) \bullet \mathbf{u}_0\right] + \mathrm{O}(\varepsilon^2). \tag{140}$$

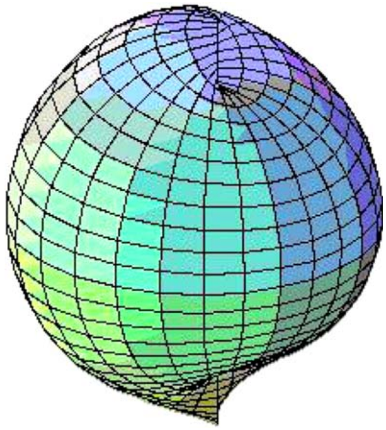

Fig. 6. Inclusion shape corresponding to Example 6.

In particular, if the applied translation is along the x_1-axis (i.e. $\mathbf{u}_0 = u_0\mathbf{e}_1$), from the formula (140), we find that

$$P_x = \frac{24\pi\mu(1-\nu)au_0}{5-6\nu}\left[1 - \varepsilon\frac{57-60\nu}{30(5-6\nu)}\right] + \mathrm{O}(\varepsilon^2), \quad P_y = \mathrm{O}(\varepsilon^2), \quad P_z = \mathrm{O}(\varepsilon^2). \tag{141}$$

Fig. 7 illustrates the dependence on the Poisson's ratio of the ratio of the magnitude of P_x (Eq. (141)) to the magnitude of P_x corresponding to the perfectly spherical inclusion for different values of the small parameterε. The effect of the Poisson's ratio is such that the magnitude of the resultant force required to produce the given translation of the inclusion decreases as the Poisson's ratio increases. For instance, at $\varepsilon = 0.5$, the magnitude of this force is about 81% of that corresponding to the perfectly spherical inclusion for $\nu = 0$ whereas that for $\nu = 0.5$ is about 77%. The author is not aware of any numerical data available in the literature, which the above results can be compared with.

On the other hand, if this inclusion is given a rotation $\boldsymbol{\omega}$ about the center of the reference sphere, the torque necessary to produce this rotation is characterized by the equation

$$\mathbf{M} = 8\pi\mu a^3\left[(1-\varepsilon)\boldsymbol{\omega} - \varepsilon\frac{1}{20}(7\mathbf{e}_1\otimes\mathbf{e}_1 - 23\mathbf{e}_2\otimes\mathbf{e}_2 + 16\mathbf{e}_3\otimes\mathbf{e}_3)\bullet\boldsymbol{\omega}\right] + \mathrm{O}(\varepsilon^2). \tag{142}$$

For instance, if the rotation $\boldsymbol{\omega}$ is given around the x_1-axis (i.e. $\boldsymbol{\omega} = \omega\mathbf{e}_1$), Eq. (142) yields:

$$M_x = 8\pi\mu a^3\omega\left(1 - \varepsilon\frac{27}{20}\right) + \mathrm{O}(\varepsilon^2), \quad M_y = \mathrm{O}(\varepsilon^2), \quad M_z = \mathrm{O}(\varepsilon^2).$$

For instance, the above formulae are indicative that for $\varepsilon = 0.5$, the magnitude of the total torque required to produce the given rotation around the x_1-axis is about 32.5% of that required for the perfectly spherical inclusion.

In another special case when the rotation $\boldsymbol{\omega}$ is given about the x_3-axis, Eq. (142) simplifies to

$$M_x = M_y = \mathrm{O}(\varepsilon^2), \quad M_z = 8\pi\mu a^3\omega\left(1 - \varepsilon\frac{9}{5}\right) + \mathrm{O}(\varepsilon^2).$$

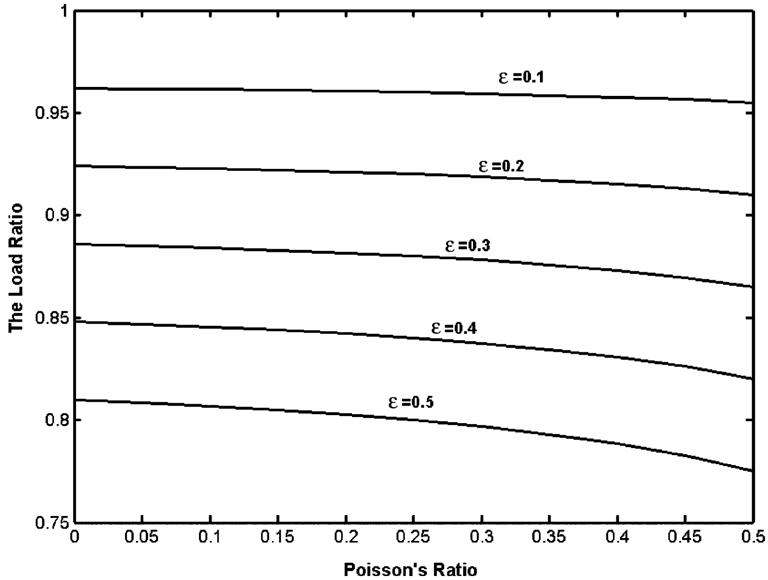

Fig. 7. Dependence of the load ratio with Poisson's ratio for different values of the small parameter Epsilon.

Example 7. Lastly, we consider an interesting case where the shape perturbation is given by

$$f(\theta,\phi) = \left\{ \begin{array}{ll} -1, & \theta_1 \leqslant \theta \leqslant \theta_2, \phi_1 \leqslant \phi \leqslant \phi_2 \\ 0, & \text{otherwise} \end{array} \right\}. \tag{143}$$

The reader would notice that this perturbation results in gouging out a small spherical element of the spherical inclusion (see Fig. 8a). The function (143) can be expanded into an infinite series of surface spherical harmonics with f_k being

$$f_k = a_{k0}P_k(\upsilon) + \sum_{i=1}^{k}[a_{ki}\cos(i\theta) + b_{ki}\sin(i\theta)]P_k^{(i)}(\upsilon), \quad \upsilon = \cos\phi,$$

where

$$a_{k0} = \frac{-(2k+1)}{4\pi}(\theta_2 - \theta_1)L_k^0,$$
$$a_{ki} = \frac{-(2k+1)(k-i)!}{2\pi i(k+i)!}[\sin i\theta_2 - \sin i\theta_1]L_k^i,$$
$$b_{ki} = \frac{(2k+1)(k-i)!}{2\pi i(k+i)!}[\cos i\theta_2 - \cos i\theta_1]L_k^i.$$

Here the following notation is introduced:

$$L_k^i(\theta_1,\theta_2,\phi_1,\phi_2) = \int_{\cos\phi_2}^{\cos\phi_1} P_k^i(\upsilon)\,\mathrm{d}\upsilon.$$

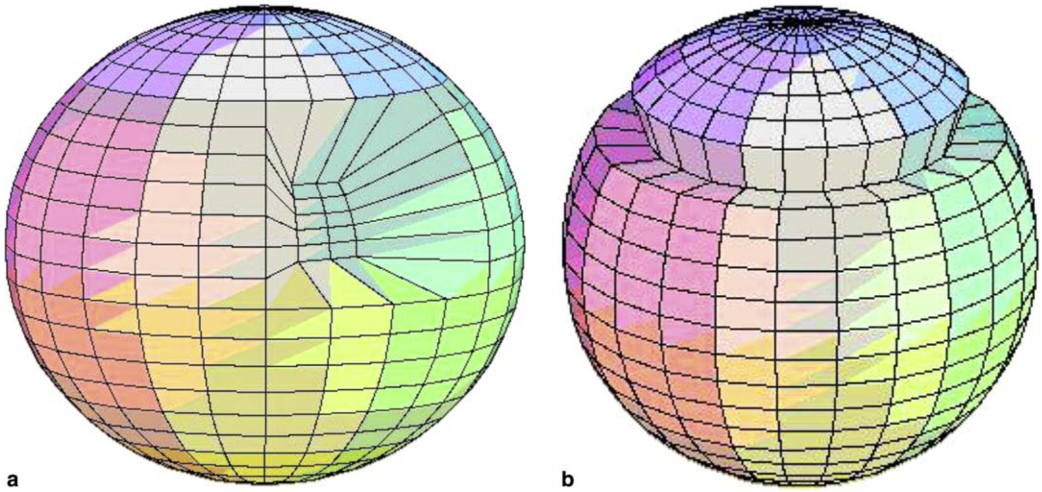

Fig. 8. (a,b) Inclusion shape corresponding to Example 7.

Since, to O(ε), the pertinent surface spherical harmonics are f_0, f_1, f_2, we only list their values below:

$$\begin{aligned}
f_0 &= \frac{-1}{4\pi}(\theta_2-\theta_1)(\cos\phi_1-\cos\phi_2),\\
f_1 &= a_{10}\cos\phi+(a_{11}\cos\theta+b_{11}\sin\theta)\sin\phi,\\
f_2 &= \frac{1}{2}a_{20}(3\cos^2\phi-1)+3(a_{21}\cos\theta+b_{21}\sin\theta)\sin\phi\cos\phi\\
&\quad+3(a_{22}\cos 2\theta+b_{22}\sin 2\theta)\sin^2\phi,
\end{aligned} \tag{144}$$

where

$$\begin{aligned}
a_{10} &= \frac{-3}{8\pi}(\theta_2-\theta_1)(\cos^2\phi_1-\cos^2\phi_2),\\
a_{11} &= \frac{-3}{8\pi}(\sin\theta_2-\sin\theta_1)\left(\phi_1-\phi_2-\frac{1}{2}\sin 2\phi_1+\frac{1}{2}\sin 2\phi_2\right),\\
b_{11} &= \frac{-3}{8\pi}(\cos\theta_2-\cos\theta_1)\left(\phi_1-\phi_2-\frac{1}{2}\sin 2\phi_1+\frac{1}{2}\sin 2\phi_2\right),\\
a_{20} &= \frac{-5}{8\pi}(\theta_2-\theta_1)(\sin^2\phi_2\cos\phi_2-\sin^2\phi_1\cos\phi_1),\\
a_{21} &= \frac{-5}{12\pi}(\sin\theta_2-\sin\theta_1)(\sin^3\phi_2-\sin^3\phi_1),\\
b_{21} &= \frac{5}{12\pi}(\cos\theta_2-\cos\theta_1)(\sin^3\phi_2-\sin^3\phi_1),\\
a_{22} &= \frac{-5}{96\pi}[\sin 2\theta_2-\sin 2\theta_1][3(\cos\phi_1-\cos\phi_2)-(\cos^3\phi_1-\cos^3\phi_2)],\\
b_{22} &= \frac{5}{96\pi}[\cos 2\theta_2-\cos 2\theta_1][3(\cos\phi_1-\cos\phi_2)-(\cos^3\phi_1-\cos^3\phi_2)].
\end{aligned} \tag{145}$$

Thus, the net force required to produce a translation of $\mathbf{u}_0$ with respect to the center of the reference sphere is given by

$$\mathbf{P} = \frac{24\pi\mu(1-v)a}{5-6v}\left[\left\{1 - \varepsilon\frac{\theta_2-\theta_1}{4\pi}(\cos\phi_1 - \cos\phi_2)\right\}\mathbf{u}_0 - \varepsilon\frac{1}{5(5-6v)}\nabla\otimes\nabla(R^2 f_2)\bullet\mathbf{u}_0\right] + \mathrm{O}(\varepsilon^2), \tag{146}$$

where $\nabla\otimes\nabla(R^2 f_2)$ assumes the following expression:

$$\begin{aligned}\nabla\otimes\nabla(R^2 f_2) &= a_{20}(3\mathbf{e}_3\otimes\mathbf{e}_3 - \widehat{\mathbf{I}}) + 3\{a_{21}(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1) + b_{21}(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2)\}\\ &\quad + 6\{a_{22}(\mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2) + b_{22}(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\}.\end{aligned} \tag{147}$$

In addition, the inclusion would exert a torque on the medium around the center of the reference sphere, which is given by

$$\mathbf{M} = \varepsilon\frac{24\pi\mu(1-v)a^2}{5-6v}(a_{11}\mathbf{e}_1 + b_{11}\mathbf{e}_2 + a_{10}\mathbf{e}_3)\times\mathbf{u}_0 + \mathrm{O}(\varepsilon^2). \tag{148}$$

Similarly, for the case of pure rotation, the net torque required to produce a rotation of $\boldsymbol{\omega}$ around the center of the reference sphere is

$$\mathbf{M} = 8\pi\mu a^3\left[\left\{1 - 3\varepsilon\frac{\theta_2-\theta_1}{4\pi}(\cos\phi_1 - \cos\phi_2)\right\}\boldsymbol{\omega} - \varepsilon\frac{3}{10}\nabla\otimes\nabla(R^2 f_2)\bullet\boldsymbol{\omega}\right] + \mathrm{O}(\varepsilon^2), \tag{149}$$

where again $\nabla\otimes\nabla(R^2 f_2)$ is given by Eq. (148). In addition, the inclusion would exert a force on the medium

$$\mathbf{P} = \varepsilon\frac{24\pi\mu(1-v)a^2}{(5-6v)}\boldsymbol{\omega}\times(a_{11}\mathbf{e}_1 + b_{11}\mathbf{e}_2 + a_{10}\mathbf{e}_3) + \mathrm{O}(\varepsilon^2). \tag{150}$$

The reader would notice that several interesting particular cases arise depending on choosing particular values of θ_1, θ_2, ϕ_1, ϕ_2. Here we carry out details of one such particular case, specifically the case where $\theta_1 = 0$, $\theta_2 = 2\pi$, which corresponds to gouging out an entire spherical strip from the reference sphere (see Fig. 8b). In this case, Eqs. (144) and (145) simplify greatly and the equations corresponding to translation, namely, Eqs. (146) and (148) reduce to

$$\begin{aligned}\mathbf{P} &= \frac{24\pi\mu(1-v)a^2}{5-6v}\left[\left\{1 - \varepsilon\frac{1}{2}(\cos\phi_1 - \cos\phi_2)\right\}\mathbf{u}_0\right.\\ &\quad\left. + \varepsilon\frac{1}{4(5-6v)}(\sin^2\phi_2\cos\phi_2 - \sin^2\phi_1\cos\phi_1)(3\mathbf{e}_3\otimes\mathbf{e}_3 - \widehat{\mathbf{I}})\bullet\mathbf{u}_0\right] + \mathrm{O}(\varepsilon^2),\\ \mathbf{M} &= -\varepsilon\frac{18\pi\mu(1-v)a^2}{5-6v}(\cos^2\phi_1 - \cos^2\phi_2)\mathbf{e}_3\times\mathbf{u}_0 + \mathrm{O}(\varepsilon^2),\end{aligned} \tag{151}$$

while those corresponding to rotation, i.e. Eqs. (149) and (150), to

$$\begin{aligned}\mathbf{M} &= 8\pi\mu a^3\left[\left\{1 - \varepsilon\frac{3}{2}(\cos\phi_2 - \cos\phi_1)\right\}\boldsymbol{\omega}\right.\\ &\quad\left. + \varepsilon\frac{3}{8}(\sin^2\phi_2\cos\phi_2 - \sin^2\phi_1\cos\phi_1)(3\mathbf{e}_3\otimes\mathbf{e}_3 - \widehat{\mathbf{I}})\bullet\boldsymbol{\omega}\right] + \mathrm{O}(\varepsilon^2),\\ \mathbf{P} &= -\varepsilon\frac{18\pi\mu(1-v)a^2}{5-6v}(\cos^2\phi_1 - \cos^2\phi_2)\boldsymbol{\omega}\times\mathbf{e}_3 + \mathrm{O}(\varepsilon^2).\end{aligned} \tag{152}$$

If, in addition to $\theta_1 = 0$, $\theta_2 = 2\pi$, we assume that $\phi_2 = \pi$, $\phi_1 = 0$, we see Eqs. (151) and (152) reduce to

$$\mathbf{P} = \frac{24\pi\mu(1-v)a(1-\varepsilon)}{5-6v}\mathbf{u}_0 + \mathrm{O}(\varepsilon^2), \quad \mathbf{M} = \mathrm{O}(\varepsilon^2), \tag{153}$$

and

$$\mathbf{M} = 8\pi\mu a^3(1 - 3\varepsilon)\boldsymbol{\omega} + O(\varepsilon^2), \quad \mathbf{P} = O(\varepsilon^2). \tag{154}$$

Note that the resulting shape perturbation produces a new perfectly spherical inclusion of radius $a(1 - \varepsilon)$ and this case was discussed in Example 1. The reader would notice that Eqs. (153) and (154) are those obtained in Example 1.

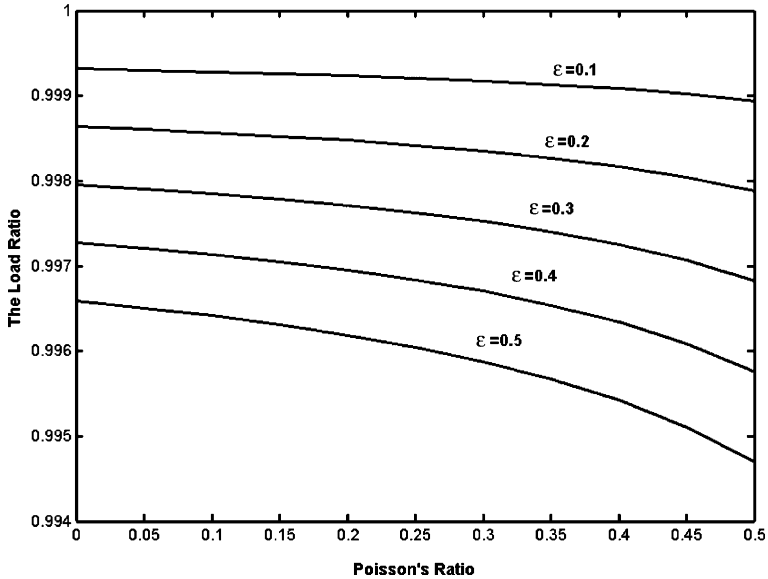

Fig. 9. Dependence of the load ratio on Poisson's ratio for different values of the small parameter.

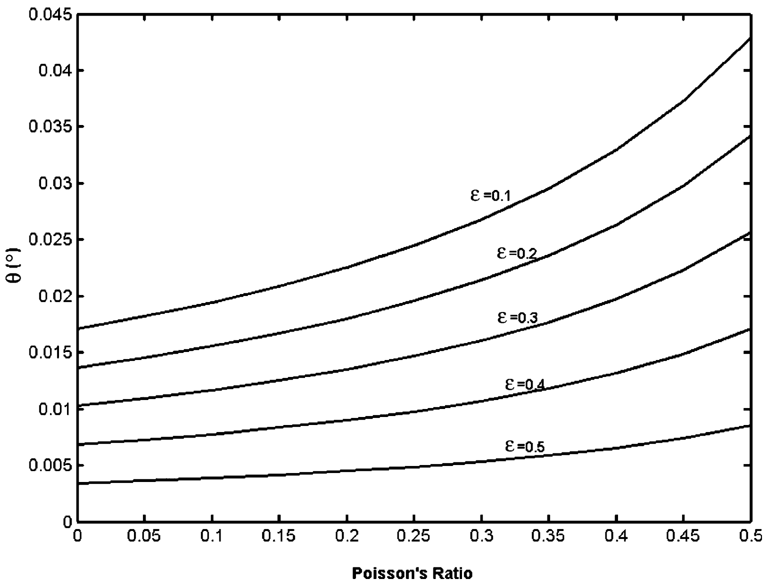

Fig. 10. Dependence of the angle Theta on Poisson's ratio for different values of the small parameter.

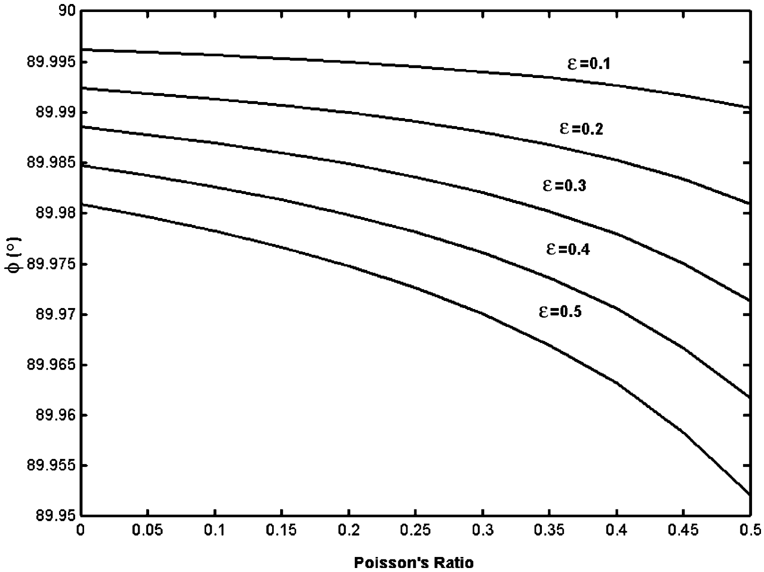

Fig. 11. Dependence of the angle Phi on Poisson's ratio for different values of the small parameter.

Finally, we have worked out a general case for the case of pure translation, specifically the case where the spherical element corresponding to $\theta_1 = \pi/4$, $\theta_2 = \pi/3$, $\phi_1 = \pi/4$, $\phi_2 = \pi/3$ is gouged out of the perfectly spherical inclusion. Fig. 9 illustrates the dependence on the Poisson's ratio of the ratio of the magnitude of the resultant force for the gouged-out inclusion to that of the perfectly spherical inclusion for different values of the small parameter ε. Figs. 10 and 11 show the dependence on the Poisson's ratio of the spherical angles characterizing the direction of the resultant force for different values of ε. The inclusion would also exert a moment on the medium which is not given here, but be easily calculated using Eq. (150).

Various other shape perturbations can be treated in a similar fashion.

8. Closure

In the present article, we have presented the solution of the problem of a nominally spherical inclusion embedded into an unbounded elastic medium and subjected to a small translation and a small rotation. Physically such translational and rotational motions of macroscopic inclusions might be attributed to certain mechanisms of diffusional migration under the action of external forces. An inclusion can also migrate in the absence of external forces. For instance, the influence of thermal fluctuations can give rise to the Brownian motion of inclusions (for detail discussions of these mechanisms, see Geguzin and Krivoglaz, 1973). To the first order in the small parameter characterizing the boundary perturbation, explicit expressions have been derived for the induced displacement field as well as for the net force and net torque required to produce the applied translation and rotation. In the special case where the elastic medium is an incompressible one, these results are consistent with those derived by Brenner (1964), for the low Reynolds number resistance of a slightly perturbed sphere to translational and rotational motions in an unbounded fluid. Although no attempt has been made to establish the convergence of the perturbation solutions, we have shown that in a particular case involving the translation of a prolate spheroidal inclusion, the first order perturbation solution is within an absolute error of 6% when compared with the exact solution of the problem. The accuracy of the perturbation solutions can be further strengthened by adding the solutions for higher order perturbation fields and using different techniques for improving the accuracy of perturbation series (see, for instance, Van Dyke, 1974). Concluding this discussion, we note that the methodology developed in the article can be applied to a number of closely-related elastostatic problems, such as the one concerning a nominally spherical hole in an elastic medium under a uniform stress field at infinity. Research in this direction is underway and will be reported in a future communication.

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Appendix A. Centroid of the nominally spherical inclusion

The centroid of the nominally spherical inclusion is determined by the formula

$$\mathbf{R}_0 = \frac{1}{V}\int_{\Omega} \mathbf{R}\, \mathrm{d}\Omega, \tag{A.1}$$

where V is the volume enclosed by the nominally spherical inclusion, i.e.

$$V = \int_{\Omega} \mathrm{d}\Omega.$$

First calculate V; we have

$$\begin{aligned} V &= \int_{\partial\Omega_1} \mathrm{d}\tau \int_0^{a(1+\varepsilon f)} R^2\, \mathrm{d}R = \frac{a^3}{3}\int_{\partial\Omega_1} (1+3\varepsilon f)\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2) = \frac{4\pi a^3}{3} + \varepsilon a^3 \int_{\partial\Omega_1} f\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2) \\ &= \frac{4\pi a^3}{3} + \varepsilon a^3 \sum_{k=0}^{\infty} \int_{\partial\Omega_1} f_k\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2) = \frac{4\pi a^3}{3} + \varepsilon 4\pi a^3 f_0 + \mathrm{O}(\varepsilon^2), \end{aligned}$$

where $\partial\Omega_1$ is the surface of a unit sphere.

We now move to evaluate the integral (A.1); since $\mathbf{e}_R$ depends on θ, ϕ only, we write

$$\begin{aligned} \mathbf{R}_0 &= \frac{1}{V}\int_{\Omega} R\mathbf{e}_R\, \mathrm{d}\Omega = \frac{1}{V}\int_{\partial\Omega_1} \mathbf{e}_R\, \mathrm{d}\tau \int_0^{a(1+\varepsilon f)} R^3\, \mathrm{d}R = \frac{a^4}{4V}\int_{\partial\Omega_1} \mathbf{e}_R(1+4\varepsilon f)\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2) \\ &= \frac{\varepsilon a^4}{V}\int_{\partial\Omega_1} \mathbf{e}_R f\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2) = \frac{\varepsilon a^4}{V}\sum_{k=0}^{\infty}\int_{\partial\Omega_1} \mathbf{e}_R f_k\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{A.2}$$

Inserting the expression (49) for $\mathbf{e}_R f_k$ into (A.2) and taking into view the fact that only 0th order surface spherical harmonics contribute to the integral in (A.2), we have

$$\mathbf{R}_0 = \frac{\varepsilon a^4}{V}\sum_{k=0}^{\infty}\int_{\partial\Omega_1} \mathbf{e}_R f_k\, \mathrm{d}\tau + \mathrm{O}(\varepsilon^2) = \frac{4\pi\varepsilon a^4}{3V}\mathbf{C}_0 + \mathrm{O}(\varepsilon^2) = \varepsilon a \nabla(R f_1) + \mathrm{O}(\varepsilon^2). \tag{A.3}$$

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